CS 188: Artificial Intelligence
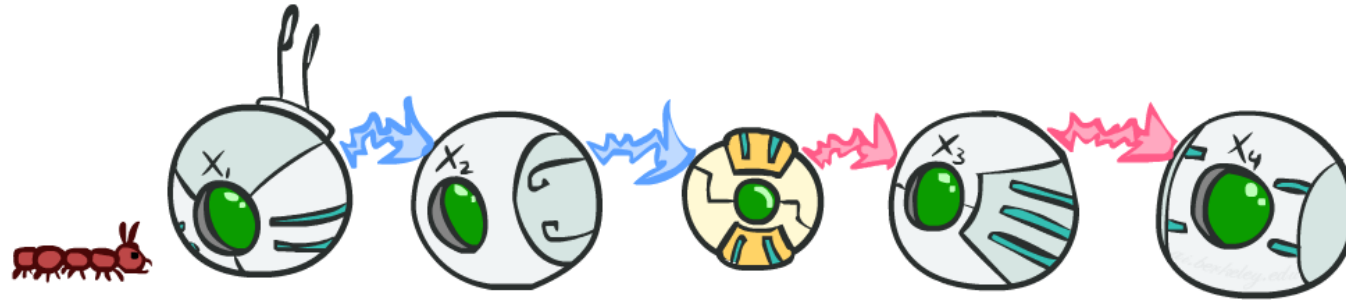
Markov Models

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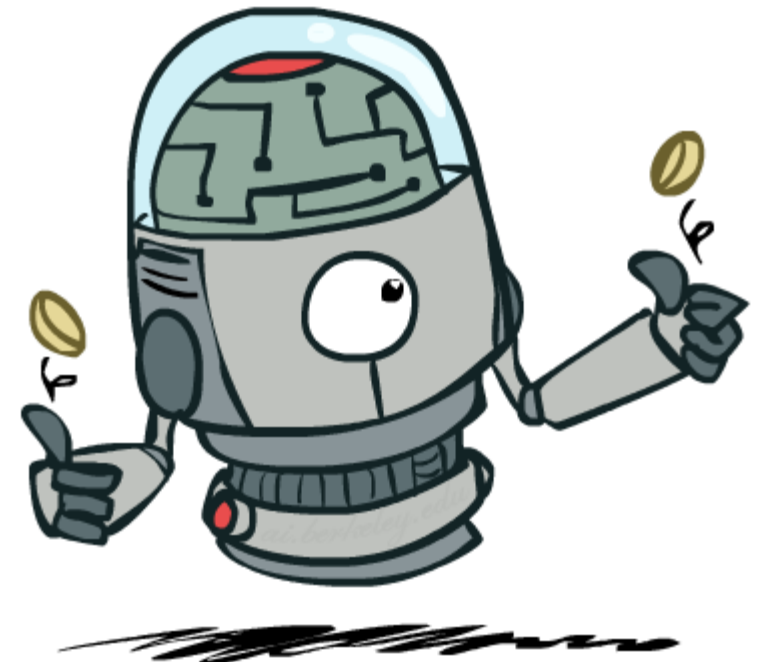
Independence

- Two variables are *independent* in a joint distribution if:

$$P(X,Y) = P(X)P(Y)$$

$$X \perp\!\!\!\perp Y$$

$$\forall x, y \, P(x,y) = P(x)P(y)$$

  - Says the joint distribution *factors* into a product of two ones
  - Usually variables aren't independent!

- Can use independence as a *modeling assumption*
  - Independence can be a simplifying assumption
  - *Empirical* joint distributions: at best "close" to independent
  - What could we assume for {Weather, Traffic, Cavity}?

- Independence is like something from CSPs: what?

Example: Independence?

$P(T)$

T	P
hot	0.5
cold	0.5

$P_1(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$P_2(T, W) = P(T)P(W)$

T	W	P
hot	sun	0.3
hot	rain	0.2
cold	sun	0.3
cold	rain	0.2

$P(W)$

W	P
sun	0.6
rain	0.4

Example: Independence

- N fair, independent coin flips:

$P(X_1)$

H	0.5
T	0.5

$P(X_2)$

H	0.5
T	0.5

. . .

$P(X_n)$

H	0.5
T	0.5

$P(X_1, X_2, \ldots X_n)$

2^n
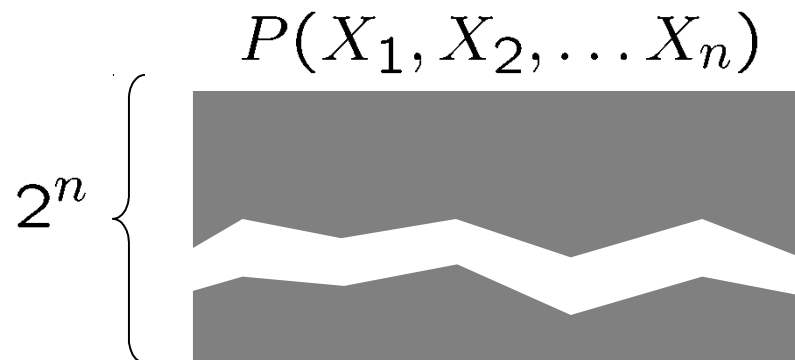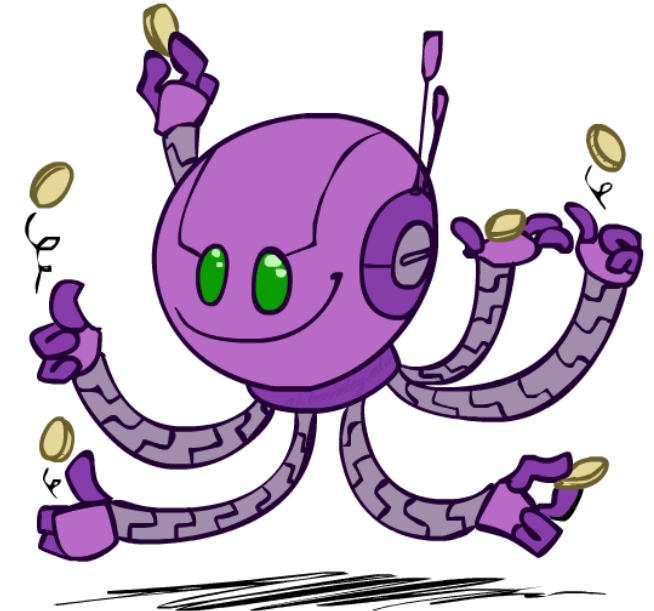
Probability Recap

- Conditional probability $$P(x|y) = \frac{P(x,y)}{P(y)}$$

- Product rule $$P(x,y) = P(x|y)P(y)$$

- Chain rule
$$
\begin{aligned}
P(X_1, X_2, \ldots X_n) &= P(X_1)P(X_2|X_1)P(X_3|X_1, X_2)\ldots \\
&= \prod_{i=1}^{n} P(X_i|X_1, \ldots, X_{i-1})
\end{aligned}
$$

- X, Y independent if and only if: $$\forall x, y : P(x,y) = P(x)P(y)$$

- X and Y are conditionally independent given Z if and only if:
$$\forall x, y, z : P(x,y|z) = P(x|z)P(y|z) \qquad X \perp\!\!\!\perp Y | Z$$

Reasoning over Time or Space

- Often, we want to reason about a sequence of observations

  - Speech recognition

  - Robot localization

  - User attention

  - Medical monitoring

- Need to introduce time (or space) into our models

Markov Models

- Value of X at a given time is called the state

$$X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow X_4 \dashrightarrow$$
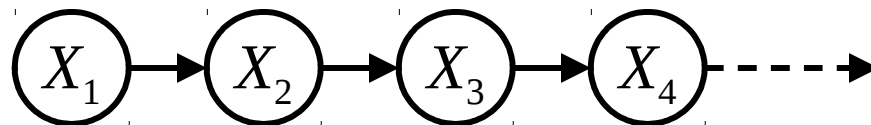
$$P(X_1) \qquad P(X_t | X_{t-1})$$

- Parameters: called transition probabilities or dynamics, specify how the state evolves over time (also, initial state probabilities)
- Stationarity assumption: transition probabilities the same at all times
- Same as MDP transition model, but no choice of action

Joint Distribution of a Markov Model

$$X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow X_4$$
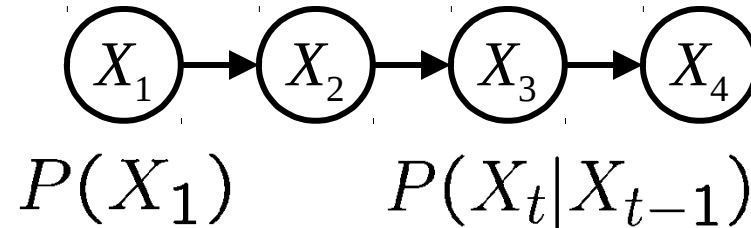
$$P(X_1) \qquad P(X_t|X_{t-1})$$

- Joint distribution:

$$P(X_1, X_2, X_3, X_4) = P(X_1)P(X_2|X_1)P(X_3|X_2)P(X_4|X_3)$$
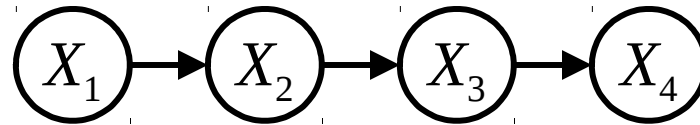
- More generally:

$$P(X_1, X_2, \ldots, X_T) = P(X_1)P(X_2|X_1)P(X_3|X_2)\ldots P(X_T|X_{T-1})$$

$$= P(X_1)\prod_{t=2}^{T} P(X_t|X_{t-1})$$

- Questions to be resolved:
  - Does this indeed define a joint distribution?
  - Can every joint distribution be factored this way, or are we making some assumptions about the joint distribution by using this factorization?

Chain Rule and Markov Models

$$X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow X_4$$

- From the chain rule, every joint distribution over X_1, X_2, X_3, X_4 can be written as:
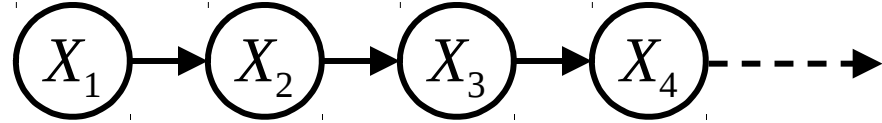
$$P(X_1, X_2, X_3, X_4) = P(X_1)P(X_2|X_1)P(X_3|X_1, X_2)P(X_4|X_1, X_2, X_3)$$

- Assuming that $X_3 \perp\!\!\!\perp X_1 \mid X_2$ and $X_4 \perp\!\!\!\perp X_1, X_2 \mid X_3$

  results in the expression posited on the previous slide:

$$P(X_1, X_2, X_3, X_4) = P(X_1)P(X_2|X_1)P(X_3|X_2)P(X_4|X_3)$$

Chain Rule and Markov Models

- From the chain rule, every joint distribution over $X_1, X_2, \ldots, X_T$ can be written as:

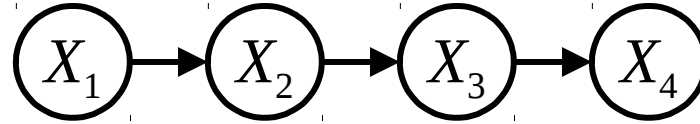$$P(X_1, X_2, \ldots, X_T) = P(X_1) \prod_{t=2}^{T} P(X_t | X_1, X_2, \ldots, X_{t-1})$$

- Assuming that for all t:

$$X_t \perp\!\!\!\perp X_1, \ldots, X_{t-2} \mid X_{t-1}$$

gives us the expression posited on the earlier slide:

$$P(X_1, X_2, \ldots, X_T) = P(X_1) \prod_{t=2}^{T} P(X_t | X_{t-1})$$

Implied Conditional Independencies

- **We assumed:** $\qquad X_3 \perp\!\!\!\perp X_1 \mid X_2 \qquad$ **and** $\qquad X_4 \perp\!\!\!\perp X_1, X_2 \mid X_3$

- **Do we also have** $X_1 \perp\!\!\!\perp X_3, X_4 \mid X_2$ **?**
  - Yes!
  - Proof:

$$P(X_1 \mid X_2, X_3, X_4) = \frac{P(X_1, X_2, X_3, X_4)}{P(X_2, X_3, X_4)}$$

$$= \frac{P(X_1)P(X_2 \mid X_1)P(X_3 \mid X_2)P(X_4 \mid X_3)}{\sum_{x_1} P(x_1)P(X_2 \mid x_1)P(X_3 \mid X_2)P(X_4 \mid X_3)}$$

$$= \frac{P(X_1, X_2)}{P(X_2)}$$

$$= P(X_1 \mid X_2)$$

Markov Models Recap

- Explicit assumption for all t : $\quad X_t \perp\!\!\!\perp X_1, \ldots, X_{t-2} \mid X_{t-1}$
- Consequence, joint distribution can be written as:

$$P(X_1, X_2, \ldots, X_T) = P(X_1)P(X_2|X_1)P(X_3|X_2)\ldots P(X_T|X_{T-1})$$

$$= P(X_1)\prod_{t=2}^{T} P(X_t|X_{t-1})$$

- Implied conditional independencies: (try to prove this!)
  - Past variables independent of future variables given the present

  i.e., if $t_1 < t_2 < t_3$ or $t_1 > t_2 > t_3$ then: $X_{t_1} \perp\!\!\!\perp X_{t_3} \mid X_{t_2}$
- Additional explicit assumption: $P(X_t \mid X_{t-1})$ is the same for all *t*
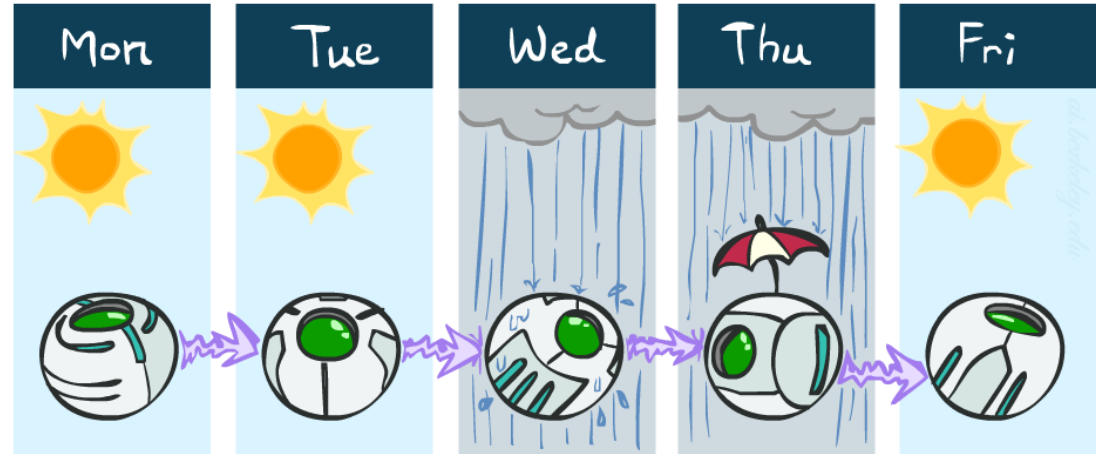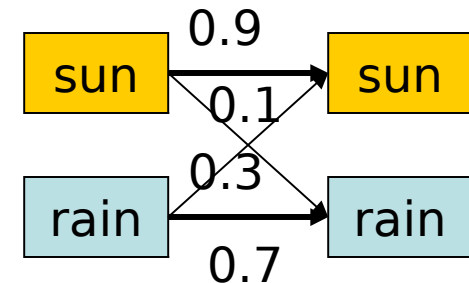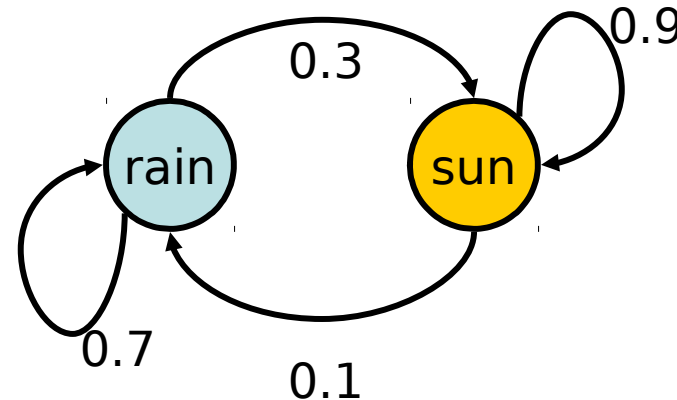
Example Markov Chain: Weather

- States: X = {rain, sun}

- Initial distribution: 1.0 sun

- CPT $P(X_t \mid X_{t-1})$:

X_{t-1}	X_t	$P(X_t\|X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7

Two new ways of representing the same CPT

0.9

0.3

0.9

rain

sun

sun → sun 0.9

0.1

rain → rain 0.3

0.7

0.7

0.1

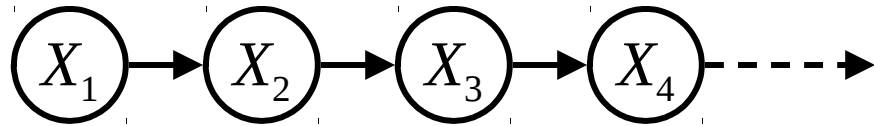# Example Markov Chain: Weather

- Initial distribution: 1.0 sun
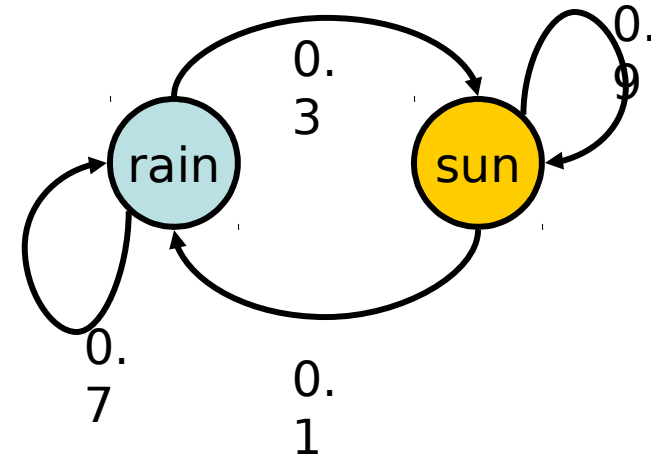
- What is the probability distribution after one step?

$$P(X_2 = \text{sun}) = \quad P(X_2 = \text{sun} | X_1 = \text{sun})P(X_1 = \text{sun}) +$$

$$P(X_2 = \text{sun} | X_1 = \text{rain})P(X_1 = \text{rain})$$

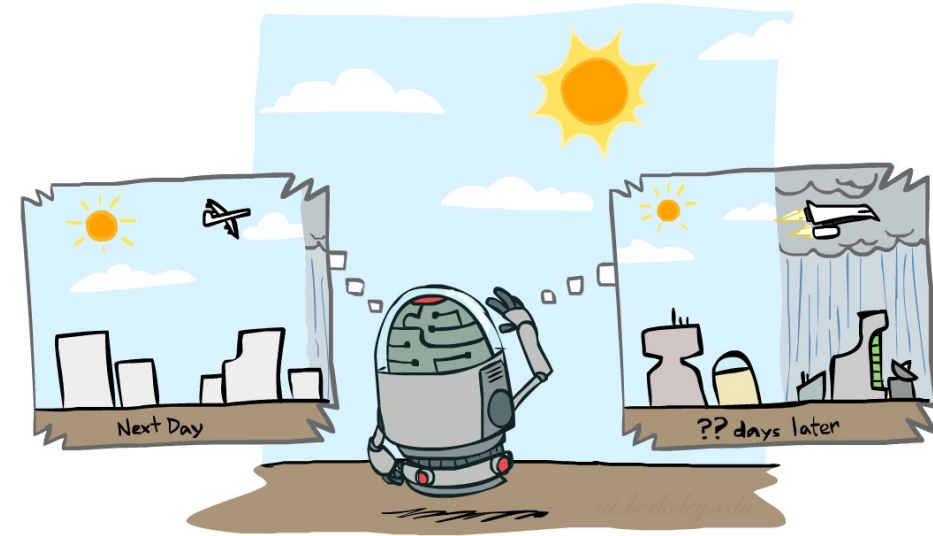$$0.9 \cdot 1.0 + 0.3 \cdot 0.0 = 0.9$$

Mini-Forward Algorithm

- Question: What's P(X) on some day t?

$$P(x_1) = \text{known}$$

$$P(x_t) = \sum_{x_{t-1}} P(x_{t-1}, x_t)$$

$$= \sum_{x_{t-1}} P(x_t \mid x_{t-1}) P(x_{t-1})$$

Forward simulation

Example Run of Mini-Forward Algorithm

- From initial observation of sun

$$\left\langle \begin{matrix} 1.0 \\ 0.0 \end{matrix} \right\rangle \quad \left\langle \begin{matrix} 0.9 \\ 0.1 \end{matrix} \right\rangle \quad \left\langle \begin{matrix} 0.84 \\ 0.16 \end{matrix} \right\rangle \quad \left\langle \begin{matrix} 0.804 \\ 0.196 \end{matrix} \right\rangle \implies \left\langle \begin{matrix} 0.75 \\ 0.25 \end{matrix} \right\rangle$$

$$\text{P}(X_1) \qquad \text{P}(X_2) \qquad \text{P}(X_3) \qquad \text{P}(X_4) \qquad \text{P}(X_\infty)$$

- From initial observation of rain

$$\left\langle \begin{matrix} 0.0 \\ 1.0 \end{matrix} \right\rangle \quad \left\langle \begin{matrix} 0.3 \\ 0.7 \end{matrix} \right\rangle \quad \left\langle \begin{matrix} 0.48 \\ 0.52 \end{matrix} \right\rangle \quad \left\langle \begin{matrix} 0.588 \\ 0.412 \end{matrix} \right\rangle \implies \left\langle \begin{matrix} 0.75 \\ 0.25 \end{matrix} \right\rangle$$

$$\text{P}(X_1) \qquad \text{P}(X_2) \qquad \text{P}(X_3) \qquad \text{P}(X_4) \qquad \text{P}(X_\infty)$$

- From yet another initial distribution P(X$_1$):

$$\left\langle \begin{matrix} p \\ 1-p \end{matrix} \right\rangle \qquad \ldots \qquad \implies \left\langle \begin{matrix} 0.75 \\ 0.25 \end{matrix} \right\rangle$$

$$\text{P}(X_1) \qquad\qquad\qquad\qquad\qquad \text{P}(X_\infty)$$

Stationary Distributions

- **For most chains:**
  - Influence of the initial distribution gets less and less over time.
  - The distribution we end up in is independent of the initial distribution

- **Stationary distribution:**
  - The distribution we end up with is called the stationary distribution P_∞ of the chain
  - It satisfies

$$P_\infty(X) = P_{\infty+1}(X) = \sum_x P(X|x)P_\infty(x)$$

Example: Stationary Distributions

- Question: What's P(X) at time t = infinity?
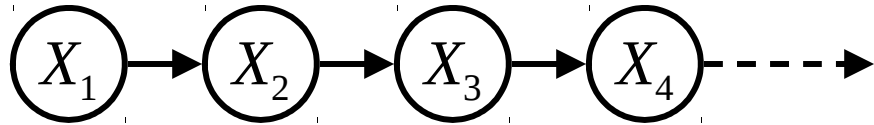
$$P_\infty(sun) = P(sun|sun)P_\infty(sun) + P(sun|rain)P_\infty(rain)$$

$$P_\infty(rain) = P(rain|sun)P_\infty(sun) + P(rain|rain)P_\infty(rain)$$

$$P_\infty(sun) = 0.9P_\infty(sun) + 0.3P_\infty(rain)$$

$$P_\infty(rain) = 0.1P_\infty(sun) + 0.7P_\infty(rain)$$

$$P_\infty(sun) = 3P_\infty(rain)$$

$$P_\infty(rain) = 1/3P_\infty(sun)$$

Also: $P_\infty(sun) + P_\infty(rain) = 1$
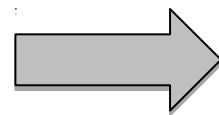
$$P_\infty(sun) = 3/4$$

$$P_\infty(rain) = 1/4$$
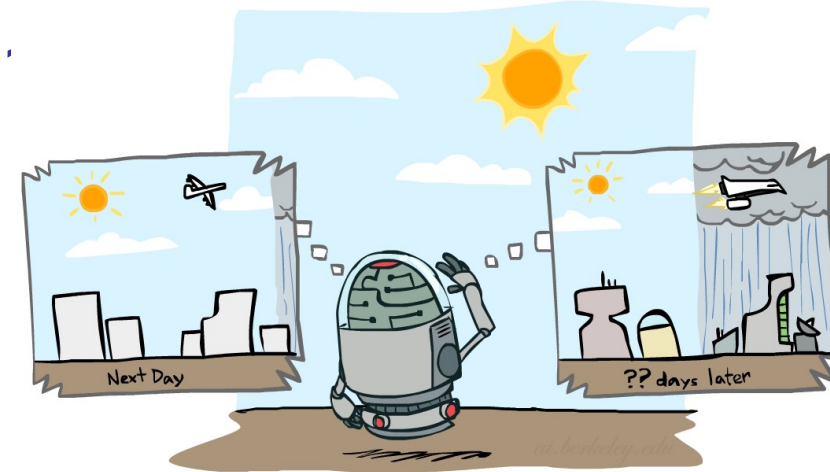
X_{t-1}	X_t	$P(X_t|X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7

Application of Stationary Distribution: Web Link Analysis

- **PageRank over a web graph**
  - Each web page is a state

  - Initial distribution: uniform over pages

  - Transitions:

    - With prob. c, uniform jump to a
      random page (dotted lines, not all shown)
    - With prob. 1-c, follow a random
      outlink (solid lines)
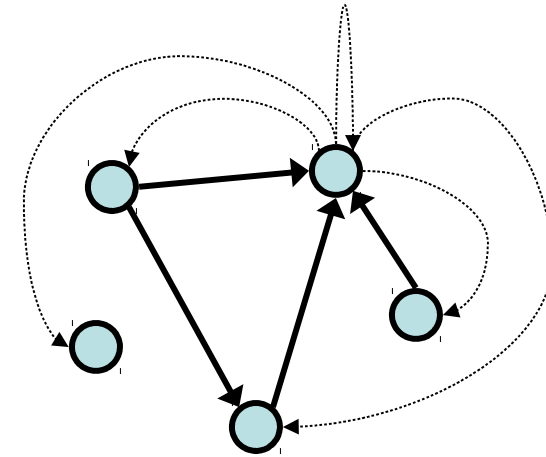
- **Stationary distribution**
  - Will spend more time on highly reachable pages
  - E.g. many ways to get to the Acrobat Reader download page
  - Somewhat robust to link spam
  - Google 1.0 returned the set of pages containing all your keywords in decreasing rank, now all search engines use link analysis along with many other factors (rank actually getting less important over time)

Next Time: Hidden Markov Models!