# Linear Regression, Neural Networks, etc.

CS444

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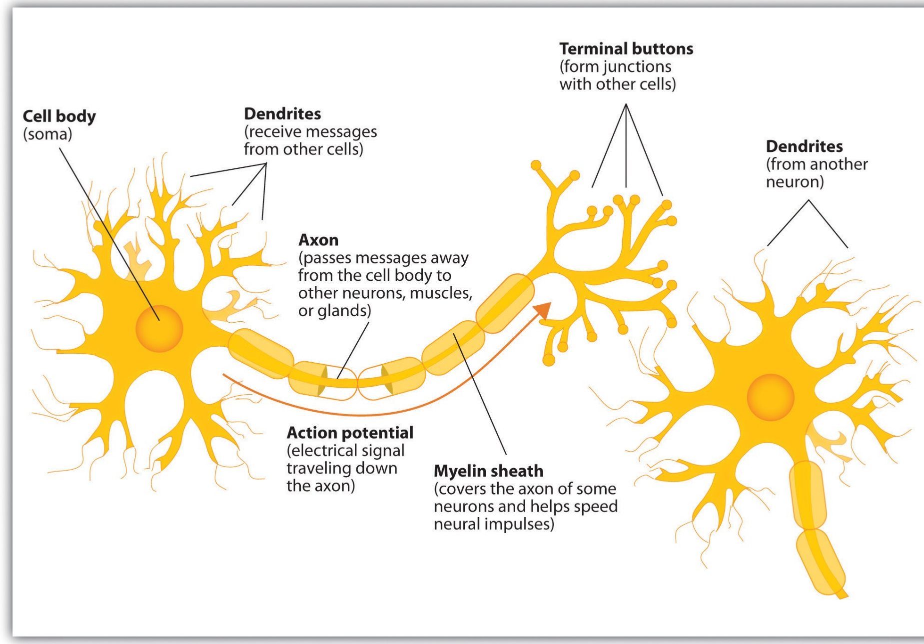
# Neurons

- Neurons communicate using discrete electrical signals called "spikes" (or action potentials).
  - Spikes travel along axons.
  - Reach axon terminals.
  - Terminals release neurotransmitters.
  - Postsynaptic neurons respond by allowing current to flow in (or out).
  - If voltage crosses a threshold a spike is created




Beginning Psychology (v. 1.0).
http://2012books.lardbucket.org/books/beginning-psychology/

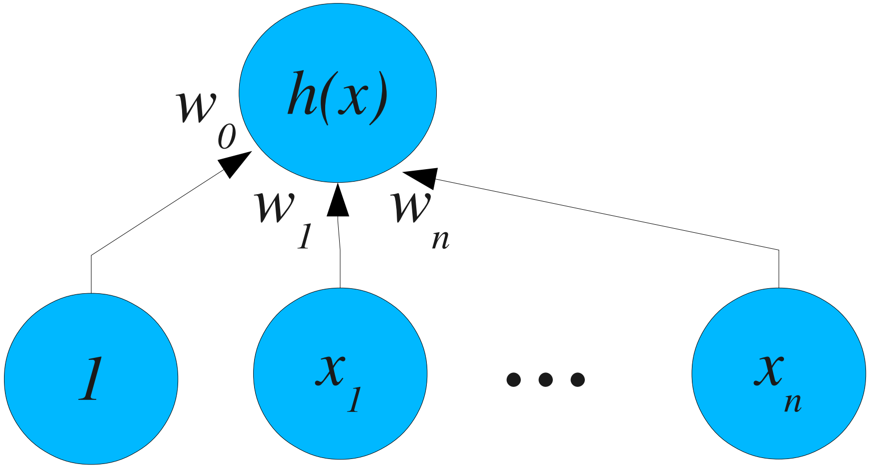
# Multivariate Linear Regression

- Multi-dimensional input vectors:

$$h(x_1, x_2, \ldots, x_n) = w_0 + w_1 x_1 + \ldots + w_n x_n$$

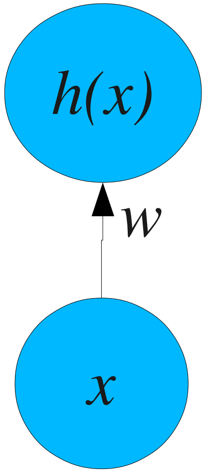
- Or:

$$h(\boldsymbol{x}) = \boldsymbol{w}^T \boldsymbol{x}$$



# Linear Regression – The Neural View

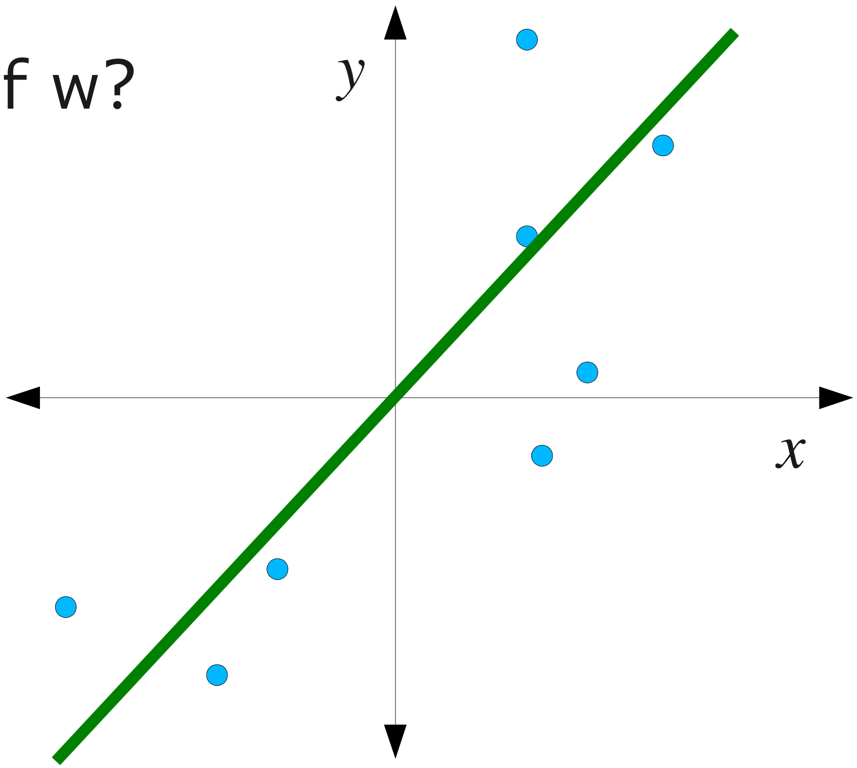
- input = $x$, desired output = $y$, weight = $w$.
- $h(x) = wx$




- We are given a set of inputs, and a corresponding set of outputs, and we need to choose $w$.
- What's going on geometrically?

# Lines

- *h(x) = wx is* the equation of a line with a y intercept of 0.
- What is the best value of w?
- How do we find it?

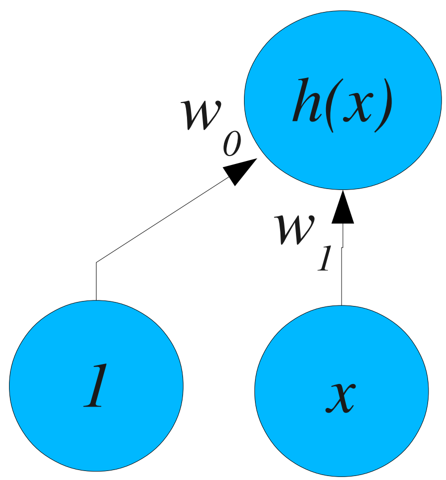


# Bias Weights

- We need to use the general equation for a line:

$$h(x) = w_1 x + w_0$$

- This corresponds to a new neural network with one additional weight, and an input fixed at 1.



# Error Metric

- Sum squared error (*y* is the desired output):

$$Error_E = \sum_{e \in E} \frac{1}{2} \left( y_e - h\left( \boldsymbol{x}_{\boldsymbol{e}} \right) \right)^2$$

- The goal is to find a *w* that minimizes *E*. How?

# Gradient Descent

http://en.wikipedia.org/wiki/File:Glacier_park1.jpg

# Gradient Descent

- One possible approach (maximization):
  1) take the derivative of the function: *f′(w)*
  2) guess a value of w : $\hat{w}$
  3) move $\hat{w}$ a little bit according to the derivative:

$$\hat{w} \leftarrow \hat{w} + \eta f'(\hat{w})$$

  4) goto 3, repeat.

# Partial Derivatives

- Derivative of a function of multiple variables, with all but the variable of interest held constant.

$$f(x, y) = x^2 + xy^2$$

$$f_x(x, y) = 2\mathrm{x} + y^2$$

OR

$$\frac{\partial f(x, y)}{\partial x} = 2\mathrm{x} + y^2$$

$$f_y(x, y) = 2\mathrm{xy}$$

OR

$$\frac{\partial f(x, y)}{\partial y} = 2\mathrm{xy}$$

# Gradient

- The gradient is just the generalization of the derivative to multiple dimensions.

$$\nabla f(\boldsymbol{w}) = \begin{bmatrix} \dfrac{\partial f(\boldsymbol{w})}{\partial w_1} \\ \dfrac{\partial f(\boldsymbol{w})}{\partial w_2} \\ \vdots \\ \dfrac{\partial f(\boldsymbol{w})}{\partial w_n} \end{bmatrix}$$

- Gradient descent update:

$$\hat{\boldsymbol{w}} \leftarrow \hat{\boldsymbol{w}} - \eta \nabla f(\hat{\boldsymbol{w}})$$

# Gradient Descent for MVLR

- Error for the multi-dimensional case:

$$Error_E(\boldsymbol{w}) = \sum_{e \in E} \frac{1}{2}\left(y_e - \boldsymbol{w}^T \boldsymbol{x_e}\right)^2$$

$$\frac{\partial\, Error_E(\boldsymbol{w})}{\partial w_i} = \sum_{e \in E} \left(y_e - \boldsymbol{w}^T \boldsymbol{x_e}\right)\left(-x_{e,i}\right)$$

$$= -\sum_{e \in E} \left(y_e - \boldsymbol{w}^T \boldsymbol{x}\right) x_{e,i}$$

- The new update rule: $w_i \leftarrow w_i + \eta \sum_{e \in E} \left(y_e - \boldsymbol{w}^T \boldsymbol{x}\right) x_{e,i}$

- Vector version: $\boldsymbol{w} \leftarrow \boldsymbol{w} + \eta \sum_{e \in E} \left(y_e - \boldsymbol{w}^T \boldsymbol{x}\right) \boldsymbol{x_e}$

# Analytical Solution

$$\boldsymbol{w} = \left(X^T X\right)^{-1} X^T y$$

- Where X is a matrix with one input per row, y the vector of target values.

# Notice that we get Polynomial Regression for Free

$$y = w_1 x^2 + w_2 x + w_0$$