Inference in first-order logic

Chapter 9

Universal instantiation (UI)

Every instantiation of a universally quantified sentence is entailed by it:

$$\frac{\forall\, v \quad \alpha}{\textsc{Subst}(\{v/g\}, \alpha)}$$

for any variable v and ground term g

E.g., $\forall\, x \;\; King(x) \land Greedy(x) \;\Rightarrow\; Evil(x)$ yields

$$\begin{array}{l} King(John) \land Greedy(John) \;\Rightarrow\; Evil(John) \\ King(Richard) \land Greedy(Richard) \;\Rightarrow\; Evil(Richard) \\ King(Father(John)) \land Greedy(Father(John)) \;\Rightarrow\; Evil(Father(John)) \\ \quad\vdots \end{array}$$

Existential instantiation (EI)

For any sentence α, variable v, and constant symbol k
that does not appear elsewhere in the knowledge base:

$$\frac{\exists\, v \;\; \alpha}{\textsc{Subst}(\{v/k\}, \alpha)}$$

E.g., $\exists\, x \;\; Crown(x) \land OnHead(x, John)$ yields

$$Crown(C_1) \land OnHead(C_1, John)$$

provided C_1 is a new constant symbol, called a Skolem constant

Another example: from $\exists\, x \;\; d(x^y)/dy = x^y$ we obtain

$$d(e^y)/dy = e^y$$

provided e is a new constant symbol

Existential instantiation contd.

UI can be applied several times to **add** new sentences;
the new KB is logically equivalent to the old

EI can be applied once to **replace** the existential sentence;
the new KB is **not** equivalent to the old,
but is satisfiable iff the old KB was satisfiable

Inference...

Bob loves someone.

If someone loves someone, then that person loves them back.

Prove:

Someone loves bob.

Unification

Suppose the KB contains just the following:

$$\forall x \;\; King(x) \land Greedy(x) \;\Rightarrow\; Evil(x)$$
$$King(John)$$
$$\forall y \;\; Greedy(y)$$
$$Brother(Richard, John)$$

We can get the inference immediately if we can find a substitution θ such that $King(x)$ and $Greedy(x)$ match $King(John)$ and $Greedy(y)$

$\theta = \{x/John, y/John\}$ works

$\textsc{Unify}(\alpha, \beta) = \theta$ if $\alpha\theta = \beta\theta$

p	q	θ
$Knows(John, x)$	$Knows(John, Jane)$	
$Knows(John, x)$	$Knows(y, OJ)$	
$Knows(John, x)$	$Knows(y, Mother(y))$	
$Knows(John, x)$	$Knows(x, OJ)$	

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$Knows(John, x)$	$Knows(y, OJ)$	$\{x/OJ, y/John\}$
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$Knows(John, x)$	$Knows(y, OJ)$	$\{x/OJ, y/John\}$
$Knows(John, x)$	$Knows(y, Mother(y))$	$\{y/John, x/Mother(John)\}$
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$Knows(John, x)$	$Knows(y, OJ)$	$\{x/OJ, y/John\}$
$Knows(John, x)$	$Knows(y, Mother(y))$	$\{y/John, x/Mother(John)\}$
$Knows(John, x)$	$Knows(x, OJ)$	$fail$

Standardizing apart eliminates overlap of variables, e.g., $Knows(z_{17}, OJ)$

Generalized Modus Ponens (GMP)

$$\frac{p_1{}',\ \ p_2{}',\ \ldots,\ p_n{}',\ \ (p_1 \wedge p_2 \wedge \ldots \wedge p_n \Rightarrow q)}{q\theta}$$ where $p_i{}'\theta = p_i\theta$ for all i

$p_1{}'$ is $King(John)$	p_1 is $King(x)$
$p_2{}'$ is $Greedy(y)$	p_2 is $Greedy(x)$
θ is $\{x/John, y/John\}$	q is $Evil(x)$
$q\theta$ is $Evil(John)$	

GMP used with KB of definite clauses (**exactly** one positive literal)
All variables assumed universally quantified

Soundness of GMP

Need to show that

$$p_1', \ \ldots, \ p_n', \ \ (p_1 \land \ldots \land p_n \Rightarrow q) \models q\theta$$

provided that $p_i'\theta = p_i\theta$ for all i

Lemma: For any definite clause p, we have $p \models p\theta$ by UI

1. $(p_1 \land \ldots \land p_n \Rightarrow q) \models (p_1 \land \ldots \land p_n \Rightarrow q)\theta = (p_1\theta \land \ldots \land p_n\theta \Rightarrow q\theta)$

2. $p_1', \ \ldots, \ p_n' \models p_1' \land \ldots \land p_n' \models p_1'\theta \land \ldots \land p_n'\theta$

3. From 1 and 2, $q\theta$ follows by ordinary Modus Ponens

Example knowledge base

The law says that it is a crime for an American to sell weapons to hostile nations. The country Nono, an enemy of America, has some missiles, and all of its missiles were sold to it by Colonel West, who is American.

Prove that Col. West is a criminal

Example knowledge base contd.

. . . it is a crime for an American to sell weapons to hostile nations:

Example knowledge base contd.

. . . it is a crime for an American to sell weapons to hostile nations:

$$American(x) \wedge Weapon(y) \wedge Sells(x, y, z) \wedge Hostile(z) \Rightarrow Criminal(x)$$

Nono . . . has some missiles

Example knowledge base contd.

... it is a crime for an American to sell weapons to hostile nations:

$American(x) \wedge Weapon(y) \wedge Sells(x, y, z) \wedge Hostile(z) \Rightarrow Criminal(x)$

Nono ... has some missiles, i.e., $\exists\, x\ Owns(Nono, x) \wedge Missile(x)$:

$Owns(Nono, M_1)$ and $Missile(M_1)$

... all of its missiles were sold to it by Colonel West

Example knowledge base contd.

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$\forall\, x\ \ Missile(x) \wedge Owns(Nono, x) \ \Rightarrow\ Sells(West, x, Nono)$

Missiles are weapons:

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$\forall\, x\;\; Missile(x) \wedge Owns(Nono, x) \;\Rightarrow\; Sells(West, x, Nono)$

Missiles are weapons:

$Missile(x) \Rightarrow Weapon(x)$

An enemy of America counts as “hostile”:

Example knowledge base contd.

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Missiles are weapons:

$Missile(x) \Rightarrow Weapon(x)$

An enemy of America counts as "hostile":

$Enemy(x, America) \;\Rightarrow\; Hostile(x)$

West, who is American ...

$American(West)$

The country Nono, an enemy of America ...

$Enemy(Nono, America)$

Forward chaining algorithm

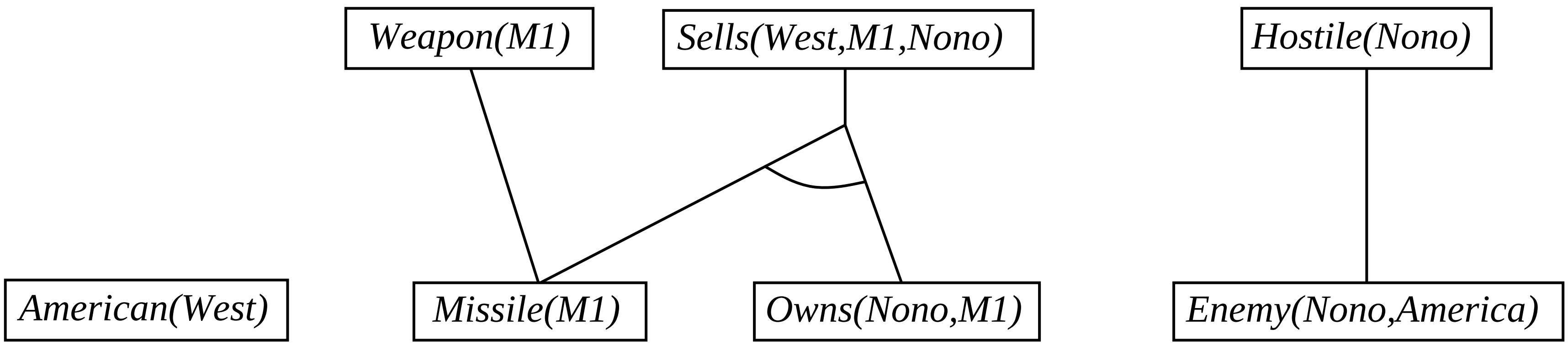
function FOL-FC-ASK(KB, α) **returns** a substitution or *false*

    repeat until *new* is empty
        $new \leftarrow \{\}$
        for each sentence r **in** KB **do**
            $(p_1 \land \ldots \land p_n \Rightarrow q) \leftarrow$ STANDARDIZE-APART(r)
            for each θ such that $(p_1 \land \ldots \land p_n)\theta = (p'_1 \land \ldots \land p'_n)\theta$
                        for some $p'_1, \ldots, p'_n$ in KB
                $q' \leftarrow$ SUBST(θ, q)
                if q' is not a renaming of a sentence already in KB or *new* **then do**
                    add q' to *new*
                    $\phi \leftarrow$ UNIFY(q', α)
                    if ϕ is not *fail* **then return** ϕ
        add *new* to KB
    return *false*

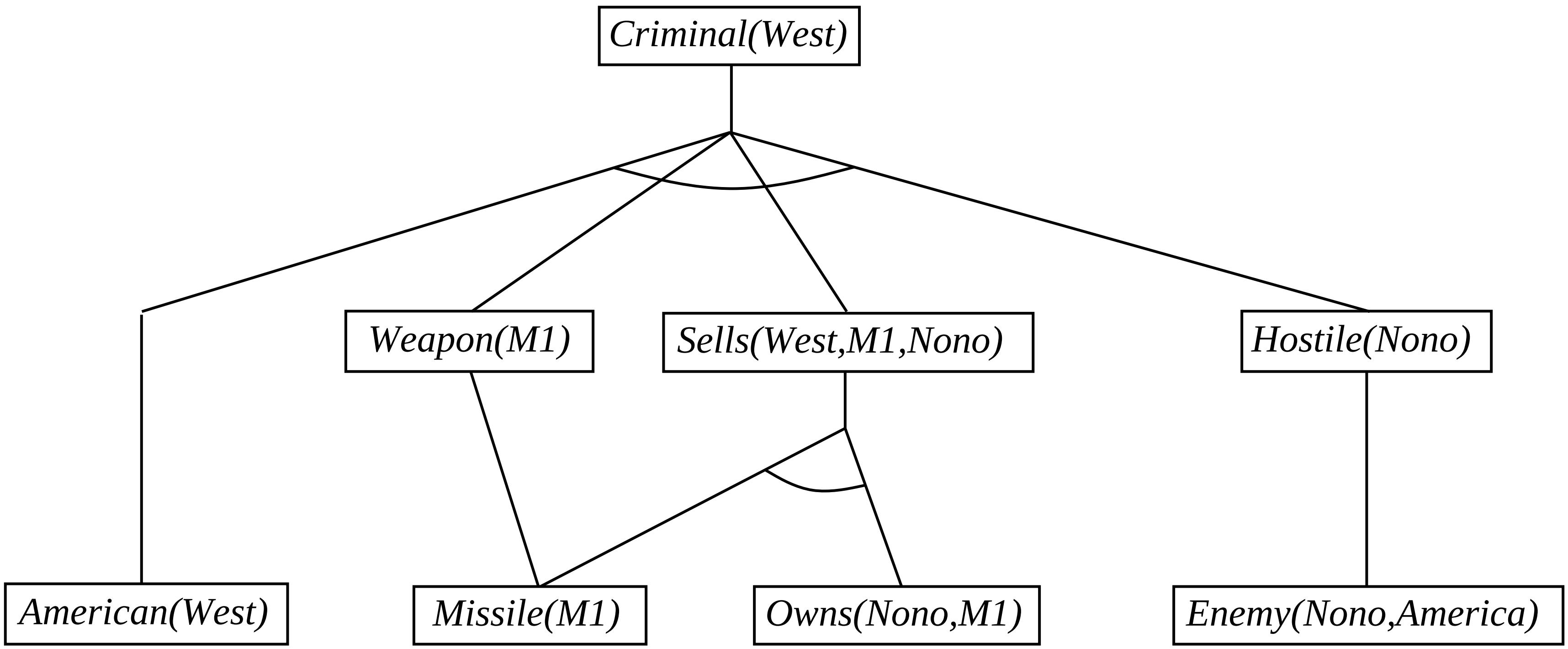
Forward chaining proof

American(West) *Missile(M1)* *Owns(Nono,M1)* *Enemy(Nono,America)*

Forward chaining proof



Forward chaining proof



Properties of forward chaining

Sound and complete for first-order definite clauses
(proof similar to propositional proof)

Datalog = first-order definite clauses + **no functions** (e.g., crime KB)
FC terminates for Datalog in poly iterations: at most $p \cdot n^k$ literals

May not terminate in general if α is not entailed

This is unavoidable: entailment with definite clauses is semidecidable

Efficiency of forward chaining

Simple observation: no need to match a rule on iteration k
if a premise wasn't added on iteration $k-1$

$\Rightarrow$ match each rule whose premise contains a newly added literal

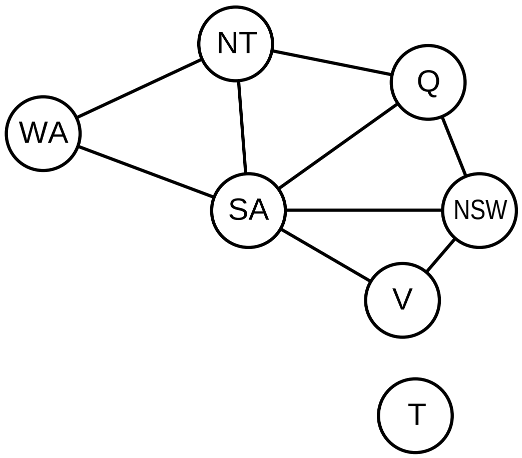
Matching itself can be expensive

Database indexing allows $O(1)$ retrieval of known facts
e.g., query $Missile(x)$ retrieves $Missile(M_1)$

Matching conjunctive premises against known facts is NP-hard

Forward chaining is widely used in deductive databases

Hard matching example



$$
\begin{aligned}
& Diff(wa,nt) \land Diff(wa,sa) \land \\
& \qquad Diff(nt,q) Diff(nt,sa) \land \\
& \qquad Diff(q,nsw) \land Diff(q,sa) \land \\
& \qquad Diff(nsw,v) \land Diff(nsw,sa) \land \\
& \qquad Diff(v,sa) \;\Rightarrow\; Colorable()
\end{aligned}
$$

$$Diff(Red,Blue) \quad Diff(Red,Green)$$
$$Diff(Green,Red) \quad Diff(Green,Blue)$$
$$Diff(Blue,Red) \quad Diff(Blue,Green)$$

$Colorable()$ is inferred iff the CSP has a solution

CSPs include 3SAT as a special case, hence matching is NP-hard

Backward chaining algorithm

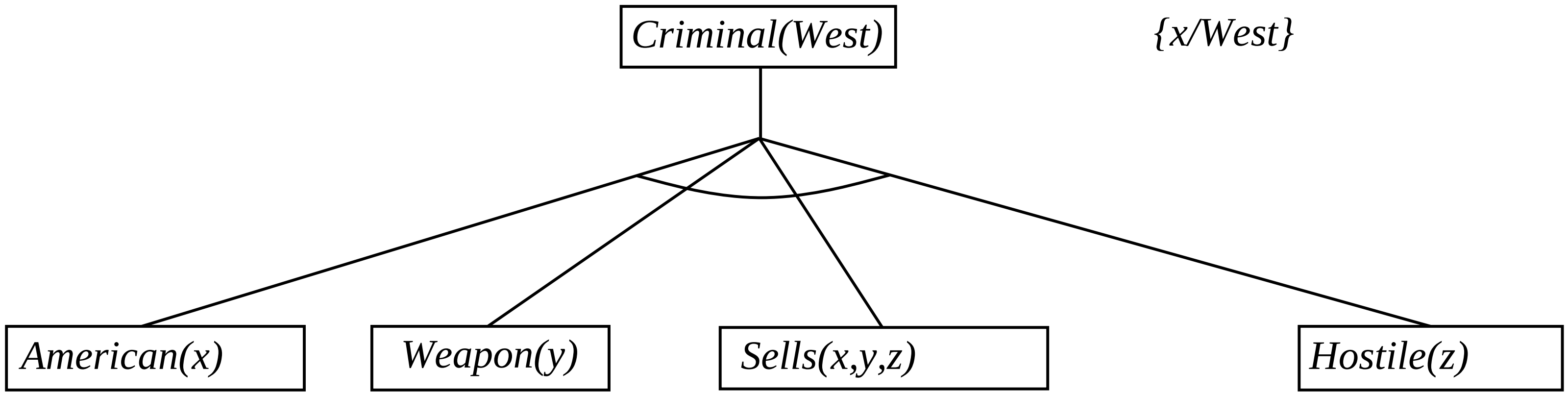
function FOL-BC-ASK($KB, goals, \theta$) **returns** a set of substitutions
  inputs: KB, a knowledge base
    $goals$, a list of conjuncts forming a query (θ already applied)
    θ, the current substitution, initially the empty substitution $\{\,\}$
  local variables: $answers$, a set of substitutions, initially empty

  if $goals$ is empty **then return** $\{\theta\}$
  $q' \leftarrow \textsc{Subst}(\theta, \textsc{First}(goals))$
  for each sentence r **in** KB
    where $\textsc{Standardize-Apart}(r) = (\, p_1 \ \wedge\ \ldots\ \wedge\ p_n \ \Rightarrow\ q)$
    and $\theta' \leftarrow \textsc{Unify}(q, q')$ succeeds
   $new\_goals \leftarrow [\, p_1, \ldots, p_n | \textsc{Rest}(goals)]$
   $answers \leftarrow \text{FOL-BC-}\textsc{Ask}(KB, new\_goals, \textsc{Compose}(\theta', \theta)) \cup answers$
  return $answers$

Backward chaining example

$\boxed{Criminal(West)}$

Backward chaining example



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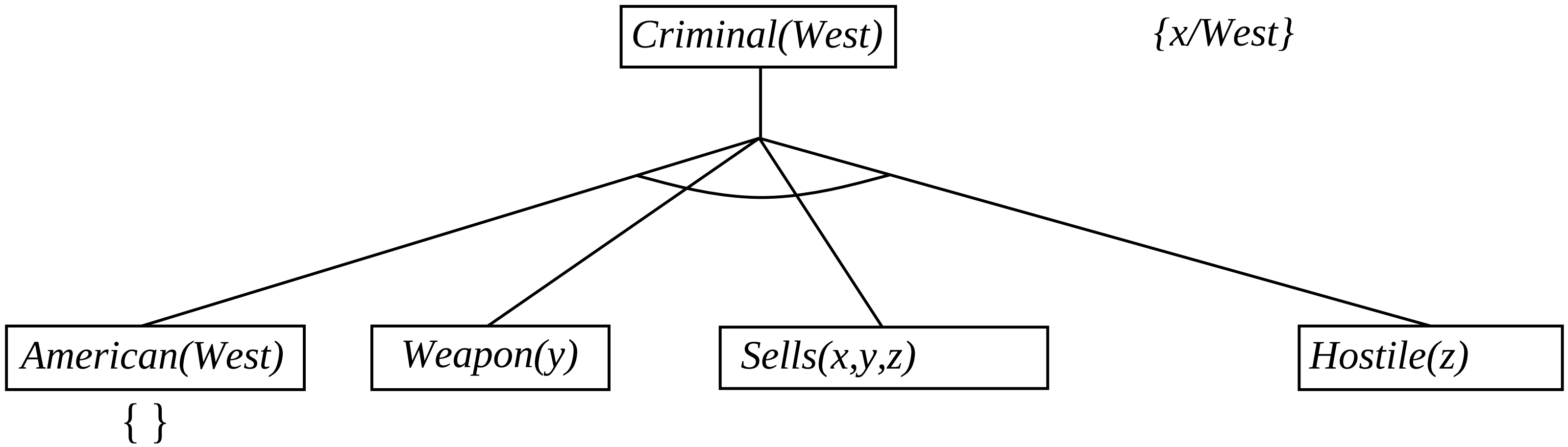

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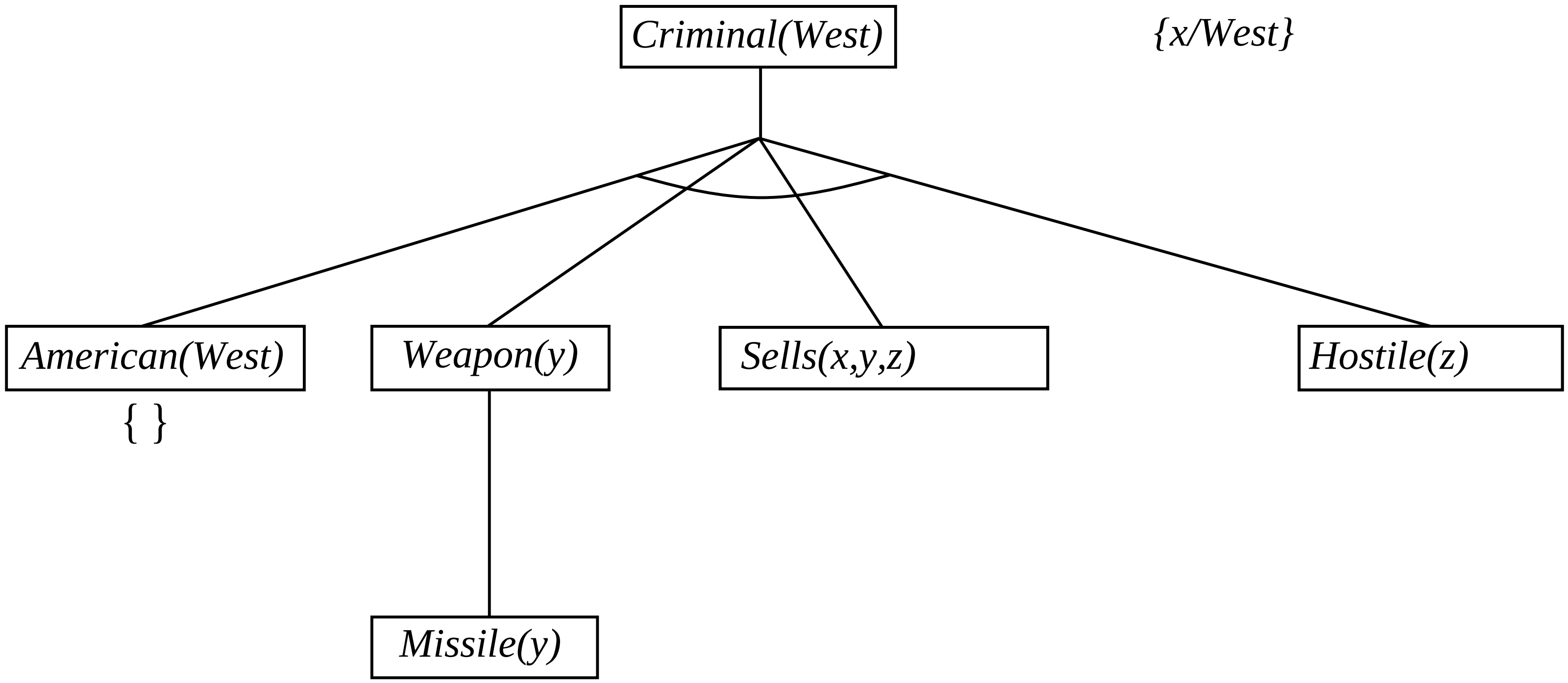

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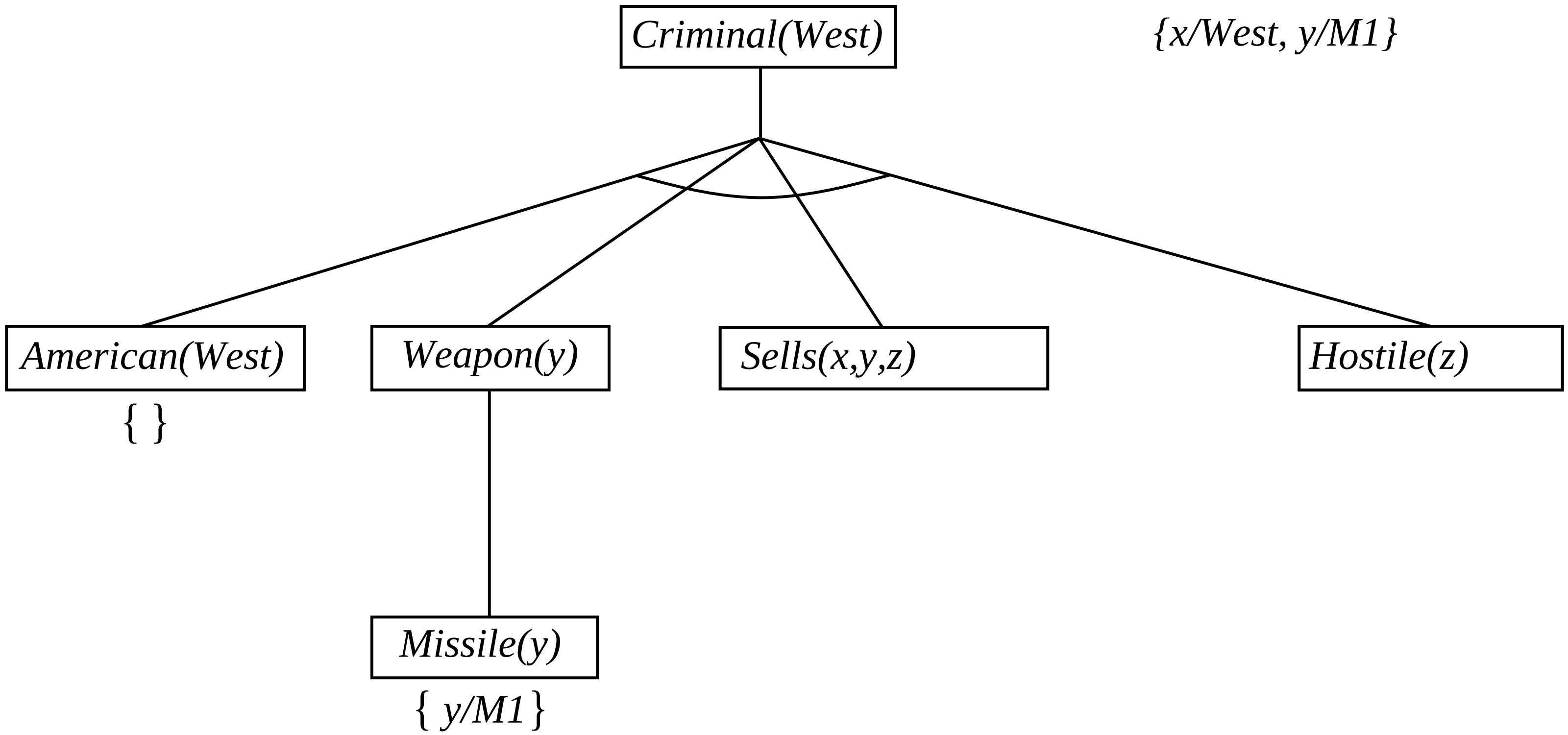

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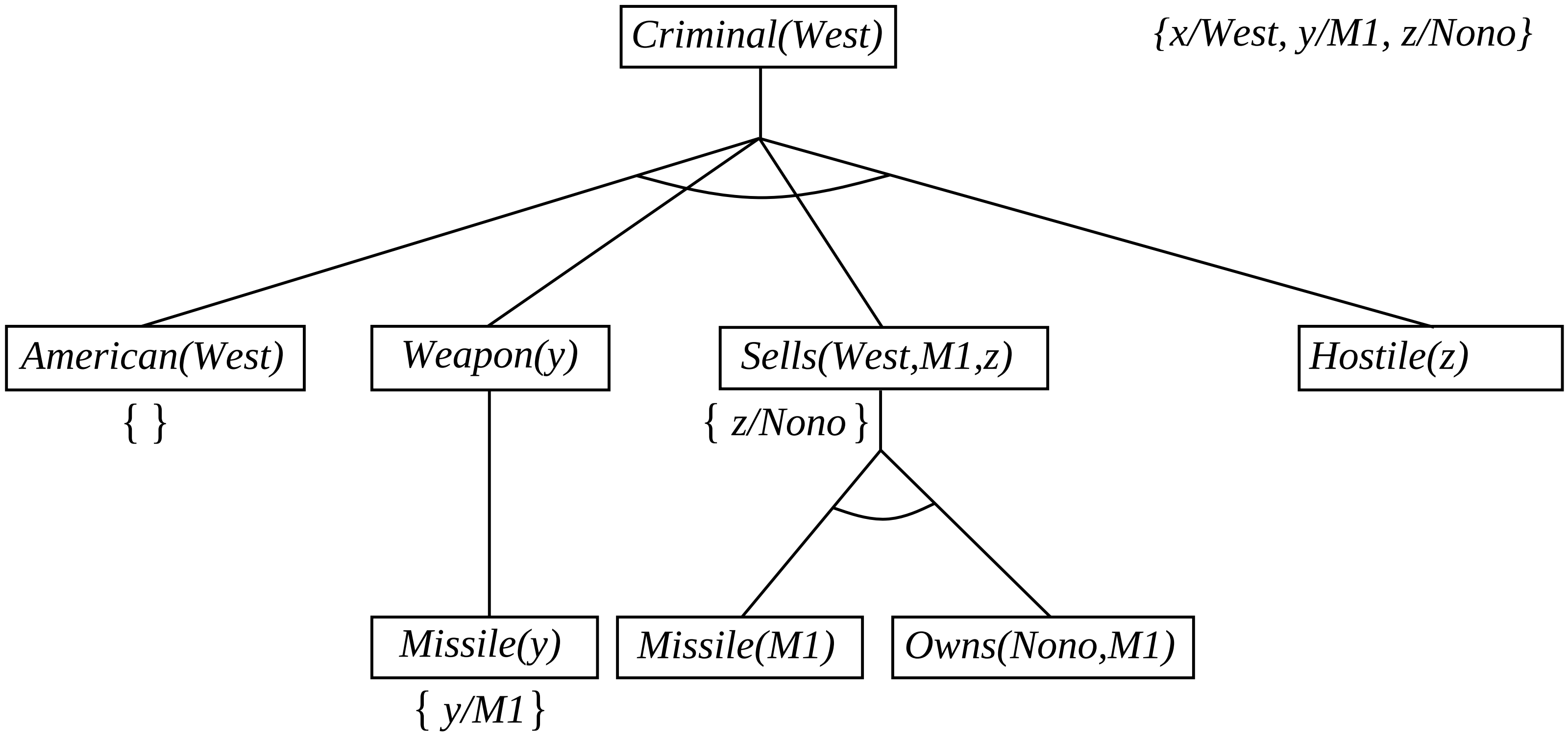

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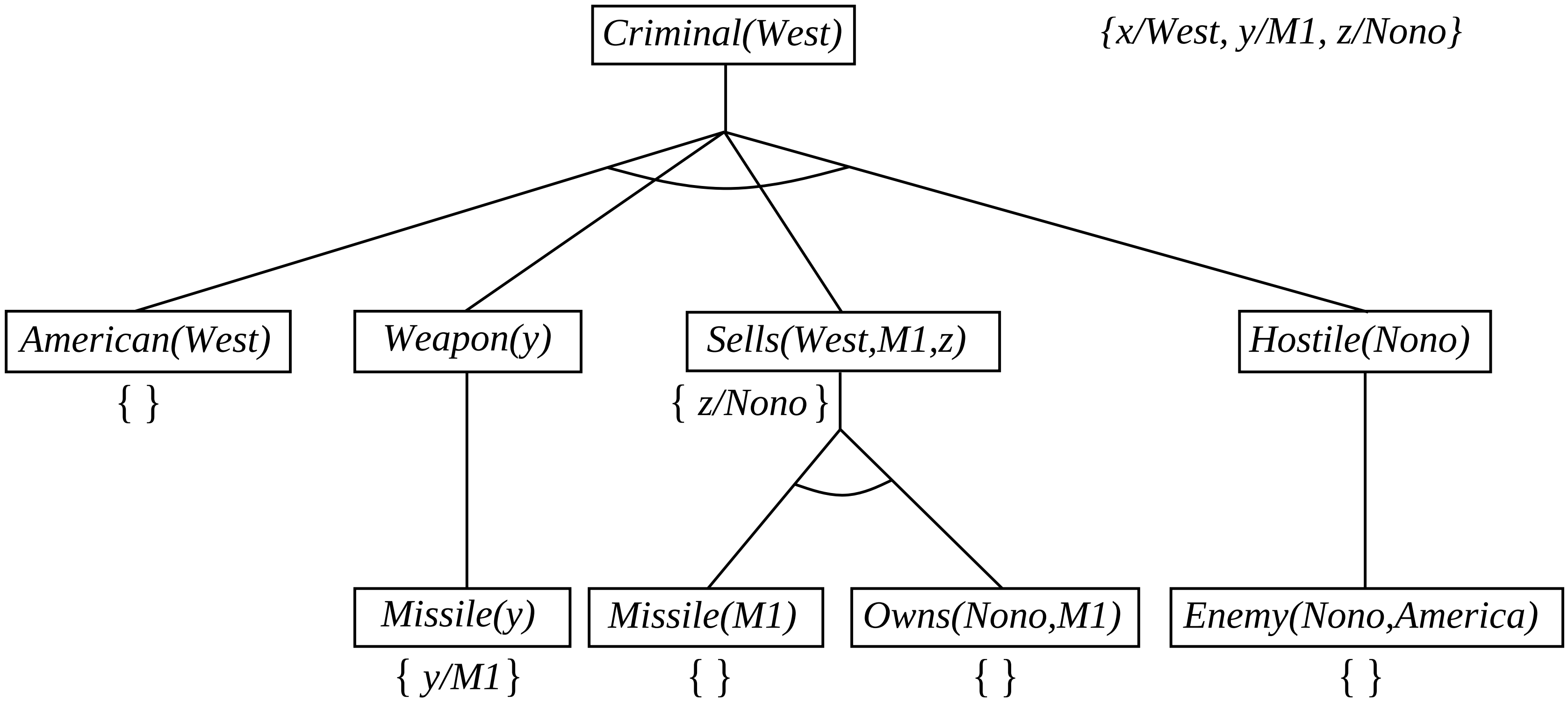

Properties of backward chaining

Depth-first recursive proof search: space is linear in size of proof

Incomplete due to infinite loops
$\Rightarrow$ fix by checking current goal against every goal on stack

Inefficient due to repeated subgoals (both success and failure)
$\Rightarrow$ fix using caching of previous results (extra space!)

Widely used (without improvements!) for logic programming

Logic programming

Sound bite: computation as inference on logical KBs

	Logic programming	Ordinary programming
1.	Identify problem	Identify problem
2.	Assemble information	Assemble information
3.	Tea break	Figure out solution
4.	Encode information in KB	Program solution
5.	Encode problem instance as facts	Encode problem instance as data
6.	Ask queries	Apply program to data
7.	Find false facts	Debug procedural errors

Should be easier to debug $Capital(NewYork, US)$ than $x := x + 2$!

Prolog systems

Basis: backward chaining with Horn clauses + bells & whistles
Widely used in Europe, Japan (basis of 5th Generation project)
Compilation techniques ⇒ approaching a billion LIPS

Program = set of clauses = `head :- literal`$_1$`, ... literal`$_n$`.`

```
criminal(X) :- american(X), weapon(Y), sells(X,Y,Z), hostile(Z).
```

Efficient unification by open coding
Efficient retrieval of matching clauses by direct linking
Depth-first, left-to-right backward chaining
Built-in predicates for arithmetic etc., e.g., `X is Y*Z+3`
Closed-world assumption ("negation as failure")
e.g., given `alive(X) :- not dead(X).`
`alive(joe)` succeeds if `dead(joe)` fails

Prolog examples

Depth-first search from a start state X:

```
dfs(X) :- goal(X).
dfs(X) :- successor(X,S),dfs(S).
```

No need to loop over S: `successor` succeeds for each

Appending two lists to produce a third:

```
append([],Y,Y).
append([X|L],Y,[X|Z]) :- append(L,Y,Z).

query:   append(A,B,[1,2]) ?
answers: A=[]    B=[1,2]
         A=[1]   B=[2]
         A=[1,2] B=[]
```

Resolution: brief summary

Full first-order version:

$$\frac{\ell_1 \vee \cdots \vee \ell_k, \qquad m_1 \vee \cdots \vee m_n}{(\ell_1 \vee \cdots \vee \ell_{i-1} \vee \ell_{i+1} \vee \cdots \vee \ell_k \vee m_1 \vee \cdots \vee m_{j-1} \vee m_{j+1} \vee \cdots \vee m_n)\theta}$$

where $\textsc{Unify}(\ell_i, \neg m_j) = \theta$.

For example,

$$\frac{\begin{array}{l}\neg Rich(x) \vee Unhappy(x) \\ Rich(Ken)\end{array}}{Unhappy(Ken)}$$

with $\theta = \{x/Ken\}$

Apply resolution steps to $CNF(KB \wedge \neg\alpha)$; complete for FOL

Conversion to CNF

Everyone who loves all animals is loved by someone:

$$\forall\, x \;\; [\forall\, y \;\; Animal(y) \;\Rightarrow\; Loves(x,y)] \;\Rightarrow\; [\exists\, y \;\; Loves(y,x)]$$

1. Eliminate biconditionals and implications

$$\forall\, x \;\; [\neg\forall\, y \;\; \neg Animal(y) \vee Loves(x,y)] \vee [\exists\, y \;\; Loves(y,x)]$$

2. Move $\neg$ inwards: $\neg\forall\, x, p \;\; \equiv \exists\, x \;\; \neg p$, $\;\; \neg\exists\, x, p \;\; \equiv \forall\, x \;\; \neg p$:

$$\forall\, x \;\; [\exists\, y \;\; \neg(\neg Animal(y) \vee Loves(x,y))] \vee [\exists\, y \;\; Loves(y,x)]$$
$$\forall\, x \;\; [\exists\, y \;\; \neg\neg Animal(y) \wedge \neg Loves(x,y)] \vee [\exists\, y \;\; Loves(y,x)]$$
$$\forall\, x \;\; [\exists\, y \;\; Animal(y) \wedge \neg Loves(x,y)] \vee [\exists\, y \;\; Loves(y,x)]$$

Conversion to CNF contd.

3. Standardize variables: each quantifier should use a different one

$$\forall x \;\; [\exists y \;\; Animal(y) \land \neg Loves(x,y)] \lor [\exists z \;\; Loves(z,x)]$$

4. Skolemize: a more general form of existential instantiation.
   Each existential variable is replaced by a Skolem function
   of the enclosing universally quantified variables:

$$\forall x \;\; [Animal(F(x)) \land \neg Loves(x,F(x))] \lor Loves(G(x),x)$$

5. Drop universal quantifiers:

$$[Animal(F(x)) \land \neg Loves(x,F(x))] \lor Loves(G(x),x)$$

6. Distribute $\land$ over $\lor$:

$$[Animal(F(x)) \lor Loves(G(x),x)] \land [\neg Loves(x,F(x)) \lor Loves(G(x),x)]$$

Resolution proof: definite clauses