First-order logic

Chapter 8

(Slides borrowed from Stuart Russel: `http://aima.eecs.berkeley.edu/slides-tex/`)

First-order logic

Whereas propositional logic assumes world contains **facts**,
first-order logic (like natural language) assumes the world contains

- Objects: people, houses, numbers, theories, Ronald McDonald, colors, baseball games, wars, centuries . . .

- Relations: red, round, bogus, prime, multistoried . . .,
  brother of, bigger than, inside, part of, has color, occurred after, owns, comes between, . . .

- Functions: father of, best friend, third inning of, one more than, end of
  . . .

Logics in general

Language	Ontological Commitment	Epistemological Commitment
Propositional logic	facts	true/false/unknown
First-order logic	facts, objects, relations	true/false/unknown
Temporal logic	facts, objects, relations, times	true/false/unknown
Probability theory	facts	degree of belief
Fuzzy logic	facts + degree of truth	known interval value

Syntax of FOL: Basic elements

Constants $KingJohn, \ 2, \ UCB, \ldots$
Predicates $Brother, \ >, \ldots$
Functions $Sqrt, \ LeftLegOf, \ldots$
Variables $x, \ y, \ a, \ b, \ldots$
Connectives $\land \ \lor \ \lnot \ \Rightarrow \quad \Leftrightarrow$
Equality $=$
Quantifiers $\forall \ \exists$

Atomic sentences

Atomic sentence $= predicate(term_1, \dots, term_n)$
or $term_1 = term_2$

Term $= function(term_1, \dots, term_n)$
or $constant$ or $variable$

E.g., $Brother(KingJohn, RichardTheLionheart)$
$> (Length(LeftLegOf(Richard)), Length(LeftLegOf(KingJohn)))$

Complex sentences

Complex sentences are made from atomic sentences using connectives

$$\neg S, \quad S_1 \wedge S_2, \quad S_1 \vee S_2, \quad S_1 \Rightarrow S_2, \quad S_1 \Leftrightarrow S_2$$

E.g. $Sibling(KingJohn, Richard) \Rightarrow Sibling(Richard, KingJohn)$
$> (1, 2) \vee \leq (1, 2)$
$> (1, 2) \wedge \neg > (1, 2)$

Truth in first-order logic

Sentences are true with respect to a model and an interpretation

Model contains ≥ 1 objects (domain elements) and relations among them

Interpretation specifies referents for
    constant symbols $\rightarrow$ objects
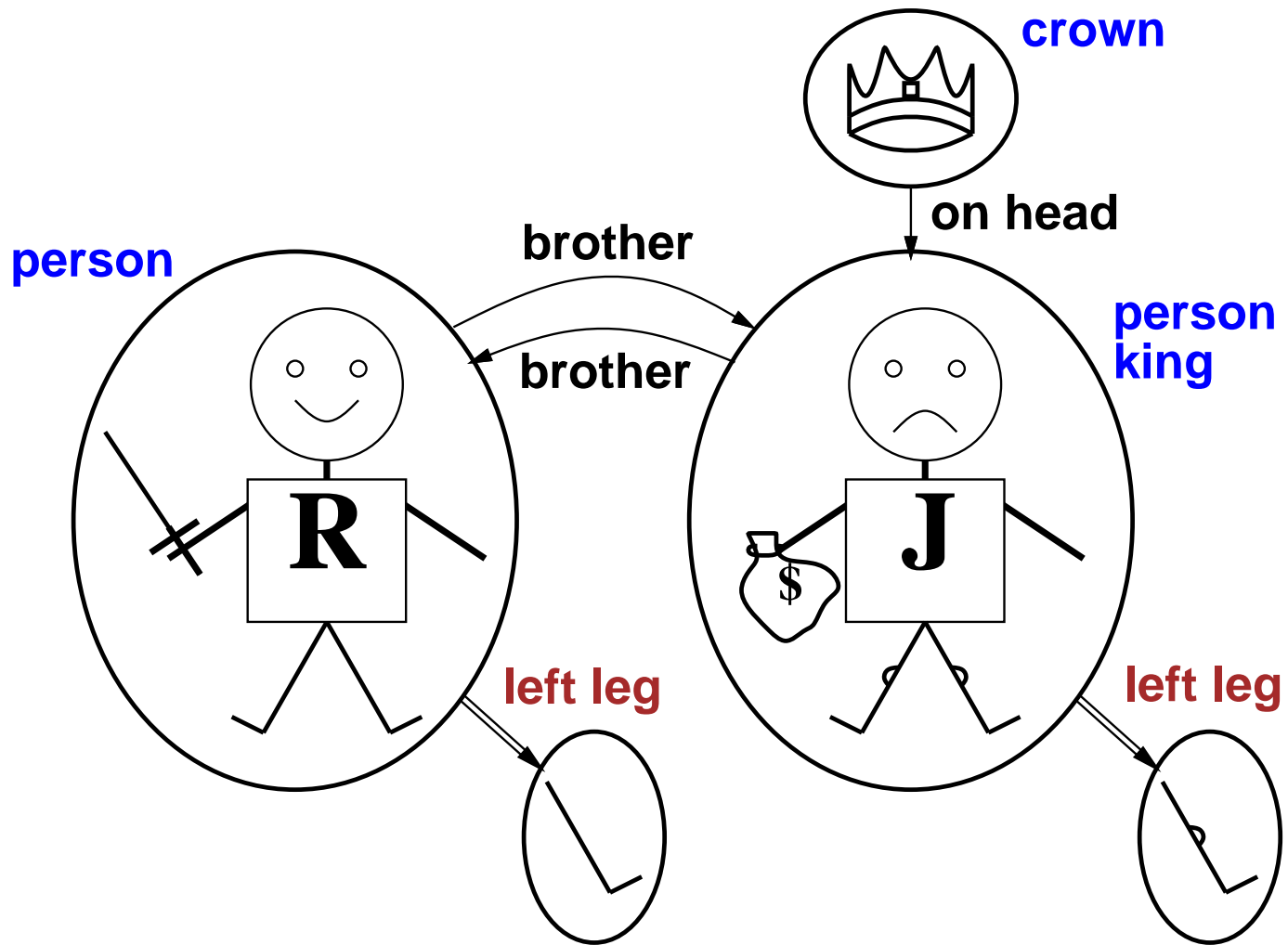   predicate symbols $\rightarrow$ relations
    function symbols $\rightarrow$ functional relations

An atomic sentence $predicate(term_1, \ldots, term_n)$ is true
iff the objects referred to by $term_1, \ldots, term_n$
are in the relation referred to by $predicate$

Models for FOL: Example

Truth example

Consider the interpretation in which
$Richard \rightarrow$ Richard the Lionheart
$John \rightarrow$ the evil King John
$Brother \rightarrow$ the brotherhood relation

Under this interpretation, $Brother(Richard, John)$ is true
just in case Richard the Lionheart and the evil King John
are in the brotherhood relation in the model

Models for FOL: Lots!

Entailment in propositional logic can be computed by enumerating models

We **can** enumerate the FOL models for a given KB vocabulary:

For each number of domain elements n from 1 to ∞
    For each k-ary predicate P_k in the vocabulary
        For each possible k-ary relation on n objects
            For each constant symbol C in the vocabulary
                For each choice of referent for C from n objects . . .

Computing entailment by enumerating FOL models is not easy!

Universal quantification

$\forall \langle variables \rangle \ \langle sentence \rangle$

Everyone at JMU is smart:
$\forall x \ \ At(x, JMU) \ \Rightarrow \ Smart(x)$

$\forall x \ \ P$ is true in a model m iff P is true with x being
each possible object in the model

Roughly speaking, equivalent to the conjunction of instantiations of P

$$(At(KingJohn, JMU) \ \Rightarrow \ Smart(KingJohn))$$
$$\wedge \ (At(Richard, JMU) \ \Rightarrow \ Smart(Richard))$$
$$\wedge \ (At(JMU, JMU) \ \Rightarrow \ Smart(JMU))$$
$$\wedge \ \dots$$

Existential quantification

$\exists \langle variables \rangle \ \langle sentence \rangle$

Someone at WVU is smart:
$\exists x \ \ At(x, WVU) \wedge Smart(x)$

$\exists x \ \ P \ \ $ is true in a model m iff P is true with x being
some possible object in the model

Roughly speaking, equivalent to the disjunction of instantiations of P

$$(At(KingJohn, WVU) \wedge Smart(KingJohn))$$
$$\vee \ (At(Richard, WVU) \wedge Smart(Richard))$$
$$\vee \ (At(WVU, WVU) \wedge Smart(WVU))$$
$$\vee \ \ldots$$

Properties of quantifiers

$\forall\,x\ \ \forall\,y$ is the same as $\forall\,y\ \ \forall\,x$

$\exists\,x\ \ \exists\,y$ is the same as $\exists\,y\ \ \exists\,x$

$\exists\,x\ \ \forall\,y$ is **not** the same as $\forall\,y\ \ \exists\,x$

$\exists\,x\ \ \forall\,y\ \ Loves(x,y)$
"There is a person who loves everyone in the world"

$\forall\,y\ \ \exists\,x\ \ Loves(x,y)$
"Everyone in the world is loved by at least one person"

Quantifier duality: each can be expressed using the other

$\forall\,x\ \ Likes(x,IceCream)\qquad \neg\exists\,x\ \ \neg Likes(x,IceCream)$

$\exists\,x\ \ Likes(x,Broccoli)\qquad \neg\forall\,x\ \ \neg Likes(x,Broccoli)$

Exercises

Brothers are siblings.

"Sibling" is symmetric.

Alice likes everyone except Bob.

There is no largest integer.

One's father is always older than one's son.

Exercises

Brothers are siblings

$\forall x, y \;\; Brother(x, y) \;\Rightarrow\; Sibling(x, y).$

Equality

$term_1 = term_2$ is true under a given interpretation
if and only if $term_1$ and $term_2$ refer to the same object

E.g., $1 = 2$ and $\forall\, x \;\times(Sqrt(x), Sqrt(x)) = x$ are satisfiable
$2 = 2$ is valid

E.g., definition of (full) $Sibling$ in terms of $Parent$:

$$\forall\, x, y \;\; Sibling(x, y) \;\Leftrightarrow\; [\neg(x = y) \wedge \exists\, m, f \;\; \neg(m = f)\, \wedge$$
$$Parent(m, x) \wedge Parent(f, x) \wedge Parent(m, y) \wedge Parent(f, y)]$$

Interacting with FOL KBs

Suppose a wumpus-world agent is using an FOL KB
and perceives a smell and a breeze (but no glitter) at $t = 5$:

$Tell(KB, Percept([Smell, Breeze, None], 5))$
$Ask(KB, \exists a \ Action(a, 5))$

I.e., does KB entail any particular actions at $t = 5$?

Answer: $Yes, \ \{a/Shoot\}$ $\leftarrow$ substitution (binding list)

Given a sentence S and a substitution σ,
$S\sigma$ denotes the result of plugging σ into S; e.g.,
$S = Smarter(x, y)$
$\sigma = \{x/Hillary, y/Bill\}$
$S\sigma = Smarter(Hillary, Bill)$

$Ask(KB, S)$ returns some/all σ such that $KB \models S\sigma$

Knowledge base for the wumpus world

"Perception"
$\forall\, b, g, t \;\; Percept([Smell, b, g], t) \;\Rightarrow\; Smelt(t)$
$\forall\, s, b, t \;\; Percept([s, b, Glitter], t) \;\Rightarrow\; AtGold(t)$

Reflex: $\forall\, t \;\; AtGold(t) \;\Rightarrow\; Action(Grab, t)$

Reflex with internal state: do we have the gold already?
$\forall\, t \;\; AtGold(t) \wedge \neg Holding(Gold, t) \;\Rightarrow\; Action(Grab, t)$

$Holding(Gold, t)$ cannot be observed
$\quad\quad \Rightarrow$ keeping track of change is essential

Deducing hidden properties

Properties of locations:
$$\forall\, x,t \;\; At(Agent, x, t) \wedge Smelt(t) \;\Rightarrow\; Smelly(x)$$
$$\forall\, x,t \;\; At(Agent, x, t) \wedge Breeze(t) \;\Rightarrow\; Breezy(x)$$

Squares are breezy near a pit:

Diagnostic rule—infer cause from effect
$$\forall\, y \;\; Breezy(y) \;\Rightarrow\; \exists\, x \;\; Pit(x) \wedge Adjacent(x, y)$$

Causal rule—infer effect from cause
$$\forall\, x,y \;\; Pit(x) \wedge Adjacent(x, y) \;\Rightarrow\; Breezy(y)$$

Neither of these is complete—e.g., the causal rule doesn't say whether squares far away from pits can be breezy

Definition for the $Breezy$ predicate:
$$\forall\, y \;\; Breezy(y) \;\Leftrightarrow\; [\exists\, x \;\; Pit(x) \wedge Adjacent(x, y)]$$

Keeping track of change

Facts hold in **situations**, rather than eternally
E.g., $Holding(Gold, Now)$ rather than just $Holding(Gold)$
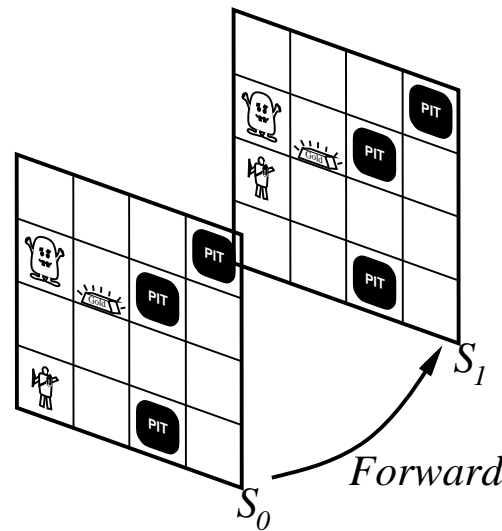
Situation calculus is one way to represent change in FOL:
       Adds a situation argument to each non-eternal predicate
       E.g., Now in $Holding(Gold, Now)$ denotes a situation

Situations are connected by the $Result$ function
$Result(a, s)$ is the situation that results from doing a in s

S_1

S_0

Forward

Describing actions I

"Effect" axiom—describe changes due to action
$$\forall s \ AtGold(s) \Rightarrow Holding(Gold, Result(Grab, s))$$

"Frame" axiom—describe **non-changes** due to action
$$\forall s \ HaveArrow(s) \Rightarrow HaveArrow(Result(Grab, s))$$

Frame problem: find an elegant way to handle non-change
      (a) representation—avoid frame axioms
      (b) inference—avoid repeated "copy-overs" to keep track of state

Qualification problem: true descriptions of real actions require endless caveats—
what if gold is slippery or nailed down or . . .

Ramification problem: real actions have many secondary consequences—
what about the dust on the gold, wear and tear on gloves, . . .

Describing actions II

Successor-state axioms solve the representational frame problem

Each axiom is "about" a **predicate** (not an action per se):

$\quad$ P true afterwards $\iff$ [an action made P true

$\qquad\qquad\qquad\qquad\quad\ \lor\quad$ P true already and no action made P false]

For holding the gold:

$\forall\, a, s\ \ Holding(Gold, Result(a, s)) \iff$
$\quad [(a = Grab \land AtGold(s))$
$\quad \lor (Holding(Gold, s) \land a \neq Release)]$

Making plans

Initial condition in KB:
$$At(Agent, [1, 1], S_0)$$
$$At(Gold, [1, 2], S_0)$$

Query: $Ask(KB, \exists s \ Holding(Gold, s))$

    i.e., in what situation will I be holding the gold?

Answer: $\{s/Result(Grab, Result(Forward, S_0))\}$

    i.e., go forward and then grab the gold

This assumes that the agent is interested in plans starting at S_0 and that S_0 is the only situation described in the KB

Making plans: A better way

Represent plans as action sequences $[a_1, a_2, \ldots, a_n]$

$PlanResult(p, s)$ is the result of executing p in s

Then the query $Ask(KB, \exists p \; Holding(Gold, PlanResult(p, S_0)))$
has the solution $\{p/[Forward, Grab]\}$

Definition of $PlanResult$ in terms of $Result$:
$\quad \forall s \; PlanResult([\,], s) = s$
$\quad \forall a, p, s \; PlanResult([a|p], s) = PlanResult(p, Result(a, s))$

Planning systems are special-purpose reasoners designed to do this type of inference more efficiently than a general-purpose reasoner

Summary

First-order logic:
- objects and relations are semantic primitives
- syntax: constants, functions, predicates, equality, quantifiers

Increased expressive power: sufficient to define wumpus world

Situation calculus:
- conventions for describing actions and change in FOL
- can formulate planning as inference on a situation calculus KB