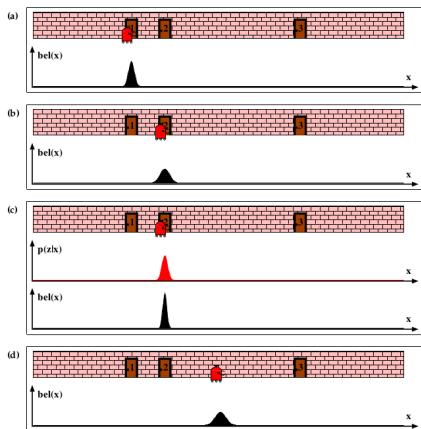


# CS354

Nathan Sprague

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# Probabilistic State Representations: Continuous



Probabilistic Robotics. Thrun, Burgard, Fox, 2005

Figure 7.6 Application of the Kalman filter algorithm to mobile robot localization. All densities are represented by unimodal Gaussians.

# Combining Evidence

- Imagine two independent measurements of some unknown quantity:
  - $x_1$  with variance  $\sigma_1^2$
  - $x_2$  with variance  $\sigma_2^2$
- How should we combine these measurements?

# Combining Evidence

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- How should we combine these measurements?
- We can take a weighted average:
  - $\hat{x} = \omega_1 x_1 + \omega_2 x_2$  (where  $\omega_1 + \omega_2 = 1$ )
- What should the weights be???

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- What should the weights be???
- We want to find weights that minimize variance (uncertainty) in the estimate:
  - $\sigma^2 = E[(\hat{x} - E[\hat{x}])^2]$

# Combining Evidence – Solution

(Derivation not shown...)

$$\hat{x} = \frac{\sigma_2^2 x_1 + \sigma_1^2 x_2}{\sigma_1^2 + \sigma_2^2}$$

$$\sigma^2 = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

## Updating an Existing Estimate

Let's reinterpret  $x_1$  to be the old state estimate and  $\sigma_1^2$  to be the variance in that estimate. Now  $x_2$  represents a new sensor reading. After some algebra...

$$\hat{x} = x_1 + \frac{\sigma_1^2(x_2 - x_1)}{\sigma_1^2 + \sigma_2^2}$$

$$\sigma^2 = \sigma_1^2 - \frac{\sigma_1^2 \sigma_1^2}{\sigma_1^2 + \sigma_2^2}$$

Let  $k = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}$ , these become...

$$\hat{x} = x_1 + k(x_2 - x_1)$$

$$\sigma^2 = \sigma_1^2 - k\sigma_1^2$$

# Vector-Valued State

- Kalman filter generalizes this to multivariate data.
- Typically the two sources of evidence are coming from:
  - Sensor
  - System Model
- We may also be combining evidence from multiple sensors
  - Sensor fusion



# Linear System Models

- State can include information other than position. E.g. velocity.
- Linear model of an object moving with a fixed velocity in 2d:
  - $x_{t+1} = x_t + \dot{x}_t dt$
  - $y_{t+1} = y_t + \dot{y}_t dt$
  - $\dot{x}_{t+1} = \dot{x}_t$
  - $\dot{y}_{t+1} = \dot{y}_t$
- $dt$  is time.
- $\dot{x}_t$  is velocity along the  $x$  axis.

# Linear System Model in Matrix Form

This is equivalent to the last slide:

$$\mathbf{x}_t = \begin{bmatrix} x_t \\ y_t \\ \dot{x}_t \\ \dot{y}_t \end{bmatrix}$$

$$\mathbf{x}_{t+1} = \begin{bmatrix} 1 & 0 & dt & 0 \\ 0 & 1 & 0 & dt \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}_t$$

# Kalman Filter

- Assumes:

- Linear state dynamics
- Linear sensor model
- Normally distributed noise in the state dynamics
- Normally distributed noise in the sensor model

- State Transition Model:

- $\mathbf{x}_t = F\mathbf{x}_{t-1} + B\mathbf{u}_{t-1} + \mathbf{w}_{t-1}$
- $\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q})$  (Normal distribution with mean 0 and covariance  $\mathbf{Q}$ )

- Sensor Model:

- $\mathbf{z}_t = H\mathbf{x}_t + \mathbf{v}_t$
- $\mathbf{v} \sim \mathcal{N}(\mathbf{0}, \mathbf{R})$

# Kalman Filter in One Slide

- Predict:

Project the state forward:

$$\hat{\mathbf{x}}_t^- = F\hat{\mathbf{x}}_{t-1} + B\mathbf{u}_{t-1}$$

Project the covariance of the state estimate forward:

$$\mathbf{P}_t^- = F\mathbf{P}_{t-1}F^T + \mathbf{Q}$$

- Correct:

Compute the Kalman gain:

$$\mathbf{K}_t = \mathbf{P}_t^- H^T (H\mathbf{P}_t^- H^T + \mathbf{R})^{-1}$$

Update the estimate with the measurement:

$$\hat{\mathbf{x}}_t = \hat{\mathbf{x}}_t^- + \mathbf{K}_t(\mathbf{z}_t - H\hat{\mathbf{x}}_t^-)$$

Update the estimate covariance:

$$\mathbf{P}_t = \mathbf{P}_t^- - \mathbf{K}_t H \mathbf{P}_t^-$$

# Extended Kalman Filter

- What if the state dynamics and/or sensor model are NOT linear?
- State Transition Model:
  - $\mathbf{x}_t = f(\mathbf{x}_{t-1}, \mathbf{u}_{t-1}) + \mathbf{w}_{t-1}$
- Sensor Model:
  - $\mathbf{z}_t = h(\mathbf{x}_t) + \mathbf{v}_t$

# Jacobian

The Jacobian is the generalization of the derivative for vector-valued functions:

$$\mathbf{J} = \frac{d\mathbf{f}}{d\mathbf{x}} = \left[ \frac{\partial \mathbf{f}}{\partial x_1} \quad \cdots \quad \frac{\partial \mathbf{f}}{\partial x_n} \right] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

$$\mathbf{J}_{ij} = \frac{\partial f_i}{\partial x_j}$$

tex borrowed from [Wikipedia](#)

# Extended Kalman Filter

- As long as  $f$  and  $h$  are differentiable, we can still use the (Extended) Kalman filter.
- Basically, we just replace the state transition and sensor update matrices with the corresponding Jacobians.