3. Path Planning

Chapter 1 explored the problem of selecting a control signal to drive a one-dimensional system into a desired goal state. This Chapter will explore the more general problem of controlling multi-dimensional systems in the presence of obstacles.

Broadly speaking, control algorithms can be broken into two categories: **reactive** and **planning-based**. In a reactive system the control signal is determined solely by comparing the current state of the system to the goal configuration. The PID controller introduced in Chapter 1 is an example of a reactive controller.

In contrast, planning-based controllers explicitly search for a sequence of steps that will move the state of the system into a goal configuration. Planning-based control tends to be more appropriate in constrained control problems such as navigating in the presence of obstacles. In these scenarios a reactive controller may drive the system into a dead end, where a planning-based controller is able to look ahead to avoid actions that won't lead to the goal. This chapter will focus on planning-based controllers.

3.1 Configuration Spaces

Robots come in all shapes and sizes, from tiny puck-shaped robots to humanoid robots with dozens of degrees of freedom. **Configuration Spaces** provide a common framework for expressing the state of a robot within its environment. A robot's configuration $\mathbf{q}$ is a vector that contains all of the information necessary to completely specify the location of a robot and all of its constituent parts. For the locomotive robot from Chapter 1, $\mathbf{q}$ would be a single scalar value indicating the robot's location on the tracks. For a humanoid robot, $\mathbf{q}$ would include all of the robot's joint angles as well as the robot's position and orientation within its workspace. The full space of possible configurations is referred to as the robot's **Configuration Space** or **C-space** and is expressed as $\mathbf{q} \in \mathcal{C}$.

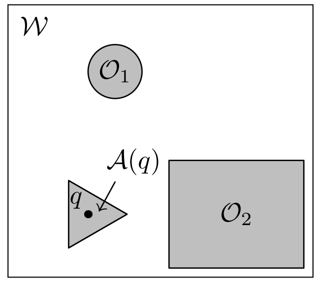


Figure 3.1: Example of a triangular robot in a planar environment.

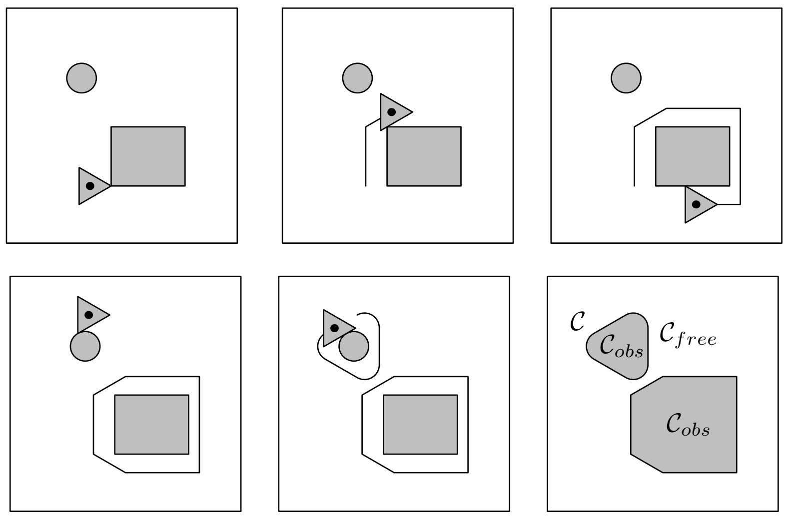


Figure 3.2: Determining $\mathcal{C}_{obs}$. The bottom-right figure illustrates $\mathcal{C}_{obs}$ and $\mathcal{C}_{free}$ for the environment shown in Figure 3.1.

While $\mathbf{q}$ may be high-dimensional, the world that the robot inhabits, $\mathcal{W}$, has only three spatial dimensions (or two if we are considering a planar path-planning problem). The mapping from a configuration $\mathbf{q}$ to the region of space occupied by a robot is denoted $\mathcal{A}(\mathbf{q}) \subset \mathcal{W}$ where $\mathcal{W} = \mathbb{R}^3$ or $\mathcal{W} = \mathbb{R}^2$.

In most workspaces there will be configurations that are impossible because they would place some part of the robot inside an obstacle. We can denote the region of space occupied by obstacles as $\mathcal{O} \subset \mathcal{W}$. For the purposes of planning it is useful to map these obstacles into the configuration space of the robot. The result is called the **C-space obstacle region** $\mathcal{C}_{obs}$ and can be expressed as $\mathcal{C}_{obs} = \{\mathbf{q} \in \mathcal{C} \mid \mathcal{A}(\mathbf{q}) \cap \mathcal{O} \neq \emptyset\}$. The unobstructed portion of of the configuration space is denoted $\mathcal{C}_{free}$ where $\mathcal{C}_{free} = \mathcal{C} - \mathcal{C}_{obs}$.

Figure 3.1 provides an example of a triangular robot in a two-dimensional workspace containing two obstacles. This robot may translate within the workspace but may not rotate. This example is atypical in that both $\mathcal{W}$ and $\mathcal{C}$ are two-dimensional. If we allowed the robot to rotate, $\mathbf{q}$ would require an additional dimension to represent the robot's orientation. The resulting configuration space would be expressed as $\mathcal{C} = \mathbb{R}^2 \times \mathbb{S}^1$, where $\mathbb{S}$ represents the set of real-valued numbers representing an angle or a point on a circle. If we were to additional appendages to the robot, like a trailer and an articulated arm, the dimensionality of $\mathcal{C}$ would increase still more.

Figure 3.2 illustrates the relationship between the shape of the robot, the locations of the obstacles and $\mathcal{C}_{obs}$. In this case, we can imagine determining $\mathcal{C}_{obs}$ by tracking $\mathbf{q}$ as we slide the robot around the boundaries of the two obstacles.

The example in Figure 3.2 is only intended to illustrate the relationship between $\mathcal{A}(\mathbf{q})$ and $\mathcal{C}_{obs}$. It doesn't represent a general algorithm for solving the problem. The problem of determining $\mathcal{C}_{obs}$ for arbitrary robots and obstacle shapes is not straightforward, particularly in higher-dimensional configuration spaces. In practice, we don't typically attempt to calculate $\mathcal{C}_{free}$ before searching for a plan. It is usually more computationally efficient to check configurations as-needed when they arise during the planning process.

One shortcut for approximating $\mathcal{C}_{obs}$ is to represent both obstacles and $\mathcal{A}(\mathbf{q})$ as spheres, or as sets of spheres. As long as the spheres are large enough to completely enclose the objects, any non-overlapping configuration is guaranteed to be in $\mathcal{C}_{free}$. This reduces collision detection to the problem of calculating distances between object centers.

Figure 3.3 provides a second example of a robot and the corresponding configuration space. This is a planar two-link articulated arm. For this robot $\mathcal{C} = \mathbb{S}^2$.

Summary of Notation

- $\mathcal{W} = \mathbb{R}^3$ (or $\mathcal{W} = \mathbb{R}^3$) - The workspace containing the robot
- $\mathbf{q}$ - The robot's configuration
- $\mathcal{C}$ - The robot's configuration space
- $\mathcal{A}(\mathbf{q}) \subset \mathcal{W}$ - The region of space occupied by robot $\mathcal{A}$ in configuration $\mathbf{q}$
- $\mathcal{O} \subset \mathcal{W}$ - The region of space occupied by obstacles
- $\mathcal{C}_{obs}$ - The subset of the configuration space that is unreachable because it involves the robot intersecting with an obstacle.
- $\mathcal{C}_{free}$ - The subset of the configuration space that is accessible to the robot

3.1.1 The Path Planning Problem

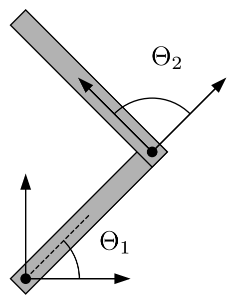
Once $\mathcal{C}_{free}$ is determined, the path-planning problem is conceptually simple. All we need to do is find a continuous path in $\mathcal{C}_{free}$ from some starting configuration $\mathbf{q}_I$ to the goal configuration $\mathbf{q}_G$. This standard formulation (sometimes called **the piano movers problem**) allows us to avoid re-thinking the path planning problem for each new robot or environment. All of robotic path planning boils down to finding a path from one point to another within a known subset of some (potentially high-dimensional) space. Figures 3.4 and 3.5 show examples of valid paths for the triangle robot and the two-link arm respectively.

3.1.2 Constraints

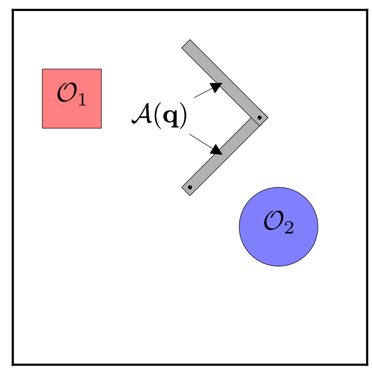
The discussion above assumes that any robot configuration is allowed, as long as the robot doesn't intersect with an obstacle. In practice, there are often additional constraints on the set of possible configurations. These constraints can be classified as **holonomic** or **non-holonomic**.

Holonomic Constraints

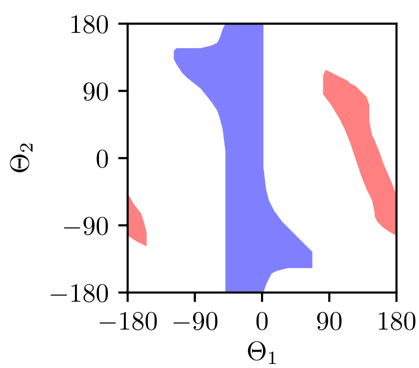
Holonomic constraints result from physical restrictions that make it impossible for the robot to enter some regions of the configuration space. For example, the elbow joint on a robotic arm may only have a 90 degree range of motion. Holonomic constraints don't significantly complicate the path planning problem: we can simply extend our notion of $\mathcal{C}_{free}$ to exclude restricted regions.



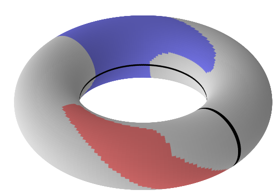
(a) 2D robot arm with two degrees of freedom. For this robot $\mathbf{q} = [\Theta_1, \Theta_2]^T$.



(b) An example of a possible configuration of the arm along with a pair of obstacles. Notice that $\mathcal{O}_2$ is close enough to the arm to prevent the first link one from making a full rotation.



(c) An illustration of $\mathcal{C}_{obs}$ for the robot configuration illustrated in 3.3b. The colors indicate configurations that intersect with the corresponding objects in 3.3b.



(d) The configuration space from 3.3c represented as a torus. The black lines are located at $\theta_1 = 180/-180$ and $\theta_2 = 180/-180$. We could imagine creating 3.3c by cutting along these lines and unwrapping the surface.

Figure 3.3: Configuration space visualizations for a planar two-link robot arm.

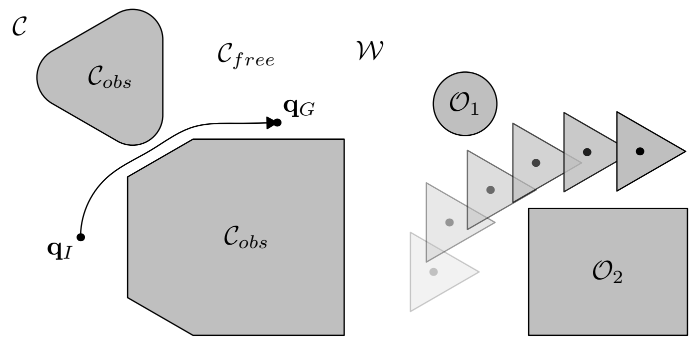


Figure 3.4: Left: A valid path from an initial configuration $\mathbf{q}_I$ to a goal configuration $\mathbf{q}_G$. Right: The robot trajectory corresponding to the indicated path.

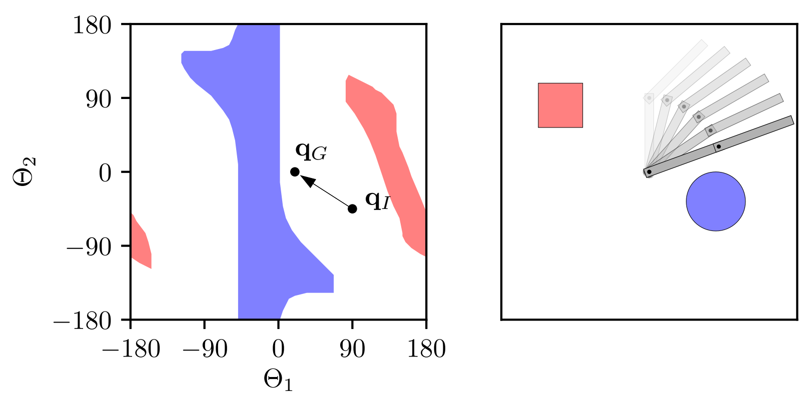


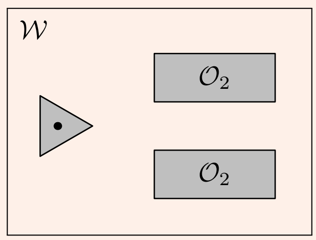
Figure 3.5: Left: A valid path from an initial configuration $\mathbf{q}_I$ to a goal configuration $\mathbf{q}_G$. Right: The robot trajectory corresponding to the indicated path.

Non-Holonomic Constraints

Non-holonomic constraints don't directly restrict which regions of the space are accessible. Instead, they restrict how the robot can move from one configuration to another. A classic example of a non-holonomic constraint is the inability of a car to slide sideways into a parking spot. There is no constraint preventing the car from being in the parking spot, but the mechanics of the vehicle prevent it from following a straight-line path to the desired configuration. Non-holonomic constraints can also arise from system dynamics: A vehicle moving at 5 miles per hour can easily make a 30 degree turn, while a vehicle moving at 50 miles per hour would roll over. Non-holonomic constraints of this sort are referred to as **kino-dynamic constraints**. Non-holonomic constraints complicate the path planning problem and require the use of specialized algorithms.

Stop and Think

3.1 Draw $\mathcal{C}_{free}$ and $\mathcal{C}_{obs}$ for the world pictured below:



Assume that this is the same non-rotating triangular robot illustrated in Figures 3.1 and 3.2. ■

3.2 The robot in Figure 3.1 has a configuration space that can be expressed as $\mathcal{C} = \mathbb{R}^2$, the robot in Figure 3.3 had a configuration space expressed as $\mathcal{C} = \mathbb{S}^2$. Imagine a more complex robot that combines attributes of these two examples. This new robot is a triangle robot that is able to translate, rotate in place, and is equipped with a two-link robotic arm. How many degrees of freedom does this robot have? How would we express configuration space for this robot? ■

3.3 Some configurations of a robotic arm may be impossible because of self-collisions. Is this an example of a holonomic or a non-holonomic constraint? ■

3.4 In the text above we used set-builder notation to express the C-obstacle region as as $\mathcal{C}_{obs} = \{\mathbf{q} \in \mathcal{C} \mid \mathcal{A}(\mathbf{q}) \cap \mathcal{O} \neq \emptyset\}$, where $\mathcal{O}$ represents the subset of $\mathcal{W}$ occupied by obstacles. What set of configurations are represented by $\{\mathbf{q} \in \mathcal{C} \mid (\mathcal{A}(\mathbf{q}) \cup \mathcal{O}) \subseteq \mathcal{O}\}$? ■

3.2 Planning Algorithms

The general path planning problem outlined above is conceptually simple but computationally difficult. Path planning has been shown to be PSPACE-Hard, making it extremely unlikely that we will ever have a polynomial time algorithm that is guaranteed to find a path if one exists. This means that practical planning algorithms necessarily involve compromises in completeness or involve attacking

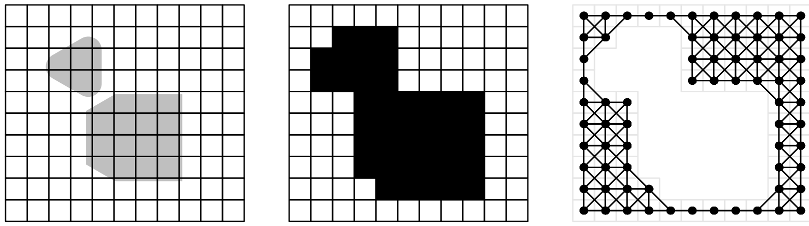


Figure 3.6: Discretizing the configuration space. The figure on the left shows a uniform grid overlaying the configuration space. In the center figure, all grid cells that overlap with $\mathcal{C}_{obs}$ have been marked as inaccessible. The figure on the right shows the state space graph that results if all accessible cells are connected to their neighbors.

simplified versions of the problem. The remainder of this chapter will explore several approaches to planning.

3.3 Grid-Based Search

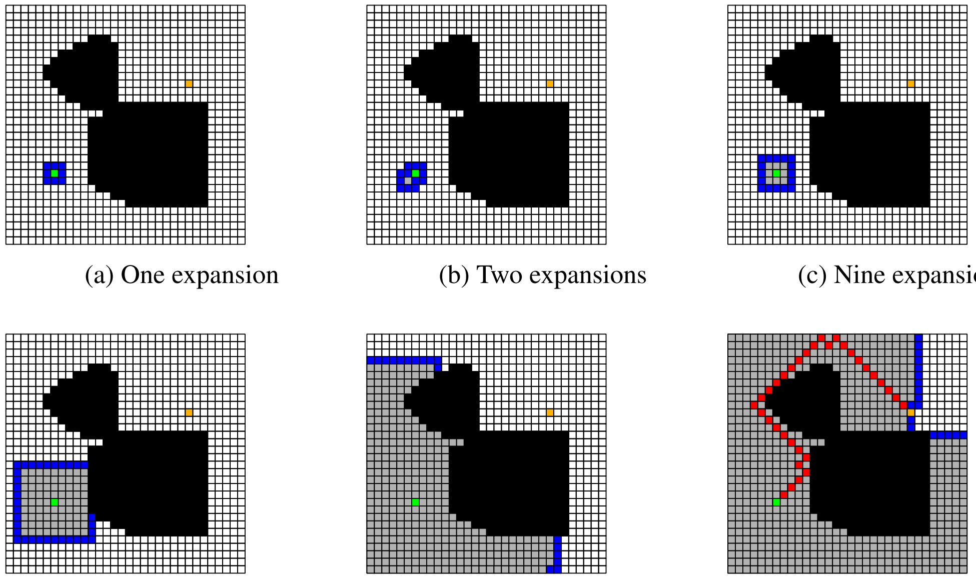
One way to simplify the planning problem is to perform a uniform discretization of the configuration space and the space of possible actions. For example, it is common to handle holonomic 2D navigation problems by overlaying the 2D configuration space with a grid. Actions are restricted to moving between four-connected or eight-connected grid cells. Figure 3.6 illustrates one possible discretization of the triangle-robot configuration space described in the previous section. The result is a graph where the vertices represent states and the edges represent actions.

Notice that the discretization in Figure 3.6 makes it impossible for the robot to find a path that passes between the two obstacles. This highlights a trade-off between the granularity of the discretization and the quality of the possible solutions. We can make our discretization arbitrarily close to the continuous version of the problem by reducing the granularity, but finer discretizations come at the cost of more grid cells which increases the cost of planning.

```python
def search(problem):
    """
    Args:
        problem: a problem instance that provides three methods:

              problem.start() -       returns the start state
              problem.goal()  -       returns the goal state
              problem.successors(s) - returns the states that are
                                      adjacent to s

    Returns:
        True if there exists a sequence of states leading from
        problem.start() to problem.goal(), or False if no such
        path exists
    """

    frontier = Collection()  # The collection of states that are
                             # currently eligible for expansion

    closed = set()           # The set of states that have already
                             # been expanded

    frontier.add(problem.start())  # Initialize the search by adding
                                   # the start state to the frontier

    while not frontier.is_empty():
        cur_state = frontier.pop()  # Select the next eligible state
                                    # to expand

        closed.add(cur_state)       # Make sure that this state can't
                                    # be added to the frontier again

        if cur_state == problem.goal():  # Success!
            return True

        else:
            # Add the neighbors of the selected state to the frontier...
            for next_state in problem.successors(cur_state):
                if (next_state not in closed and
                        next_state not in frontier):
                    frontier.add(next_state)

    return False   # No path was found!
```

Listing 1: Basic graph search algorithm.

Once a search problem has been formulated as a graph, we can use standard graph-search algorithms to find a permissible path to the goal. Listing 1 presents the basic graph search algorithm. The algorithm works outward from the start state, maintaining a collection of states on the frontier of the region we have already searched. At each iteration, we select a state from the frontier and add its neighbors back



(a) One expansion (b) Two expansions (c) Nine expansions

(d) 81 expansions (e) 350 expansions (f) 641 expansion

Figure 3.7: BFS search example. The cell containing the start state is colored green and the goal state is colored gold. States in the frontier are shown in blue. States in the closed set are shown in gray. The final path is shown in red.

into the frontier. This process continues until the goal state is selected from the frontier.

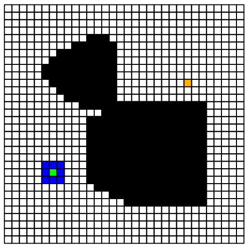
Selecting and processing a state from the frontier is referred to as *expansion*. Maintaining a *closed set* allows us to avoid re-processing states that have already been expanded. Each time a state is expanded it is added to the closed set. Once a state is closed, the check on line 39 ensures that it will never be added back into the frontier. This check prevents the algorithm from considering paths containing cycles.

The collection type used to store the frontier determines the order that the states are expanded during search. When a queue is used, the algorithm in Listing 1 performs a breadth-first-search (BFS), while the use of a stack results in depth-first-search (DFS).

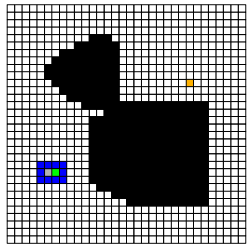
For search problems with uniform step costs, BFS is both **complete** and **optimal**. A complete search algorithm is guaranteed to find a path if one exists. An optimal search algorithm is guaranteed to find the lowest-cost path. BFS is optimal in the sense that it is guaranteed to find the path with the fewest possible steps. By storing the frontier in a FIFO queue we ensure that all paths of length n are explored before we explore any paths of length $n+1$.

In contrast, DFS is neither complete nor optimal. In the case of an infinite graph, DFS could potentially perform infinite expansions away from the goal state, even if the goal could be reached within only a small number of steps. The advantage of DFS is that it requires less memory to store the frontier. The number of nodes in the frontier grows linearly for DFS but it may grow exponentially for BFS, depending on the structure of the search problem.

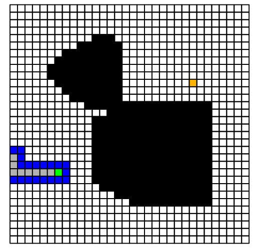
Figure 3.7 illustrates a BFS search in our triangle-world workspace. Figure 3.8 illustrates a DFS search.



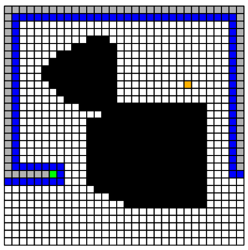
(a) One expansion



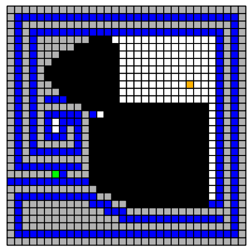
(b) Two expansions



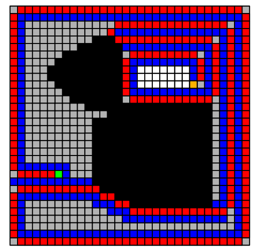
(c) Nine expansions



(d) 81 expansions



(e) 350 expansions



(f) 477 expansion

Figure 3.8: DFS search example.

3.3.1 Returning A Path

The careful reader will notice that the algorithm in Listing 1 doesn't actually return a path to the goal. It only determines whether such a path exists. In order to reconstruct the path we need to maintain an auxiliary data structure that stores backward references from each state to the state that precedes it. Once the goal is reached, the path can be reconstructed by following the backward references from the goal back to the start state and then reversing the steps. Listing 2 shows a possible Python implementation of the necessary data type and an appropriately updated version of the algorithm from Listing 1.

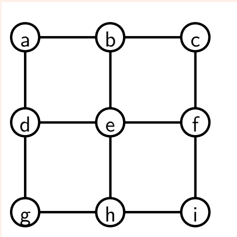
The containment check on line 40 of Listing 2 requires some clarification. The goal here is to avoid exploring multiple paths that pass through the same state. If the frontier already contains a node for a state, we don't want to add another node that represents a different route to that state. This means that the test on line 40 is *not* checking to see if the indicated Node is in the frontier, it is checking to see if any Node in the frontier contains `new_state`. Removing this check would not impact the correctness of the algorithm, but it would significantly impact efficiency for problems that allow many different paths to each state.

```python
class Node:
    """The Node class stores backward references from each state
       encountered in the search to a Node representing the
       state that preceded it.
    """
    def __init__(self, state, parent_node):
        self.state = state
        self.parent = parent_node


def search(problem):
    """
    Input: problem - a problem instance that provides three methods:

              problem.start() -       returns the start state
              problem.goal()  -       returns the goal state
              problem.successors(s) - returns the states that are
                                      adjacent to s

    Returns: A sequence of states leading from problem.start() to
             problem.goal(), or None if no path exists
    """
    frontier = Collection()
    closed = set()

    frontier.add( Node(problem.start(), None) )

    while not frontier.is_empty():
        cur_node  = frontier.pop()
        cur_state = cur_node.state
        closed.add(cur_state)

        if cur_state == problem.goal():
            return  construct_path(cur_node)  # path ending at this node

        else:
            for next_state in problem.successors(cur_state):
                next_node = Node(next_state, cur_node)
                if (next_state not in closed and
                        next_node  not in frontier): # See text.
                    frontier.add( next_node )

    return  None
```

Listing 2: Graph search algorithm updated to return a path. Steps that differ from Listing 1 are highlighted in yellow. The key difference is that the frontier stores Node objects rather than states.

Stop and Think

3.5 Consider the following graph:



Complete the table below to show the state of search algorithm 2 after each iteration of a DFS search starting at state g with the goal at state a. The tuples in the Frontier column represent Node objects where the first entry is the state and the second entry is the state associated with the parent node. Assume that the loop on line 37 accesses neighbors in alphabetical order. The first three steps are completed for you.

Expansions	Chosen	Frontier	Closed Set
0	–	$\langle g, - \rangle$	–
1	g	$\langle d, g \rangle$, $\langle h, g \rangle$	$\{g\}$
3	h	$\langle d, g \rangle$, $\langle e, h \rangle$, $\langle i, h \rangle$	$\{g, h\}$
3			
4			
5			
6			
7			

What is the final path discovered by DFS? (You should be able to reconstruct it by working backward through the parent entries starting from the point where a is selected from the frontier.)

3.6 Repeat the previous exercise using a BFS search.

3.7 The algorithm in Listing 1 is described as a graph search algorithm, but the graph is implicit: graph edges are essentially generated as needed through calls to `problem.successors`. Why might this approach make more sense than explicitly creating a complete graph structure before executing a search?

3.8 What if we knew that the state space being searched is actually a tree rooted at the start state? How would this allow us to simplify the search algorithm in Listing 1? Can you think of a path-planning problem that would have a tree-structured search space?

3.3.2 Dijkstra's Algorithm

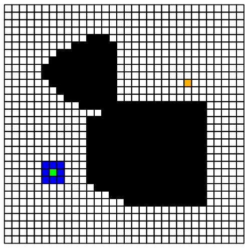
```
class Node:
    def __init__(self, state, parent_node, step_cost):
        self.state = state
        self.parent = parent_node
        self.path_cost = parent_node.path_cost + step_cost

def dijkstra(problem):
    frontier = PriorityQueue()
    closed = set()

    start_node = Node(problem.start(), None, 0.0)
    frontier.add(start_node, 0.0)

    while not frontier.is_empty():
        cur_node = frontier.pop()
        cur_state = cur_node.state
        closed.add(cur_state)

        if cur_state == problem.goal():
            return construct_path(cur_node)
        else:
            for next_state in problem.successors(cur_state):
                cost = problem.cost(cur_state, next_state)
                next_node = Node(next_state, cur_node, cost)
                if next_state not in closed:
                    frontier.add(next_node, next_node.path_cost)
    return None
```

Listing 3: Dijkstra's algorithm. Steps that differ from Listing 2 are highlighted. The key difference is that the frontier is represented by a Priority Queue that is ordered by the total cost to reach the state.

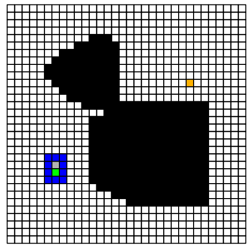
In many path-planning problems there is a notion of path cost that is distinct from the number of edges in the path. In the case of 2D navigation, the cost of a path might be the total distance traveled, the time required to reach the goal, or the amount of energy expended by the robot.

For example, if we want to find shortest paths in the grid navigation problem from Figure 3.6 we should assign weights to the edges in proportion to the distance that the robot must travel to move from cell to cell. This means that diagonal steps must have a higher weight to reflect the fact that the robot moves farther. An optimal solution to this weighted version of the problem will represent the shortest path in terms of distance traveled. Notice that the path discovered by BFS in Figure 3.7 is optimal in terms of the number of steps taken, but it is clearly not optimal in terms of distance.

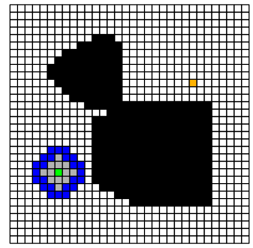
With some slight modification, the algorithm in Listing 2 can be updated to take path costs into account. Listing 3 shows the updated algorithm. The main differences are that the Node implementation has been updated to store the total path cost required to reach the corresponding state, and the frontier is now represented by a Priority Queue ordered by path cost. The Priority Queue ensures that Nodes are



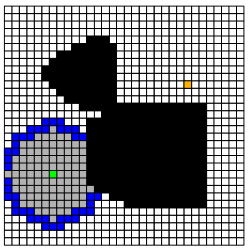
(a) One expansion



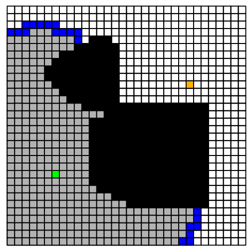
(b) Two expansions



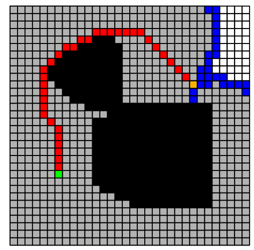
(c) 13 expansions



(d) 100 expansions



(e) 350 expansions



(f) 674 expansion

Figure 3.9: Dijkstra's algorithm search example

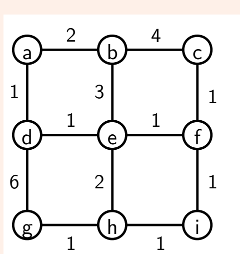
expanded in strictly increasing order of path cost. Given that all edges must have positive weights, each time an expansion occurs, all of the nodes added back to the frontier are guaranteed to have a higher cost than the node that was expanded. This means that when a node representing the goal state is expanded it must represent the lowest-cost path to the goal.

Notice that the algorithm in Listing 3 does not check if a state is already on the frontier before adding the new node for that state. From the point of view of correctness, this is fine: There is no danger of finding a longer path to the goal before a shorter path is discovered since the lower-cost node will be expanded first. However, it is more efficient to implement the frontier so that any higher-cost node is replaced when a lower-cost node is added for the same state. This prevents the wasted effort of expanding a higher-cost node (representing a longer path to the same state) after a lower-cost node.

Figure 3.9 shows an example of applying Dijkstra's algorithm to the triangle navigation problem. The resulting path is optimal for the given discretization.

Stop and Think

3.9 Consider the following weighted graph:



Complete the table below to show the state of Dijkstra's algorithm after each expansion. Again the start state is g and the goal state is a. Break priority ties using alphabetical order. I.e. in the case of a tie, state a will selected before state b. The tuples in the Frontier column represent Node objects where the first entry is the state, the second entry is the state associated with the parent node and the third entry is the path cost to reach the state. The first three steps are completed for you.

Expansions	Chosen	Frontier	Closed Set
0	–	$\langle g, -, 0\rangle$	–
1	g	$\langle d, g, 6\rangle$, $\langle h, g, 1\rangle$	$\{g\}$
3	h	$\langle d, g, 6\rangle$, $\langle e, h, 3\rangle$, $\langle i, h, 2\rangle$	$\{g, h\}$
3			
4			
5			
6			
7			
8			

What final path is returned? What is the path cost? ■

3.10 Compare 3.7 to 3.9. Why is the frontier more or less square in one and round in the other? ■

3.11 Take a look at the true, non-discretized, configuration space from Figure 3.1. What would the shortest path look like in that space? Why is it different from the path shown in figure 3.9? ■

3.12 Is it guaranteed that there will always be a unique minimum-cost path? If not, how could the algorithm in Listing 3 be updated to return all such paths? ■

3.3.3 A*

All of the graph search algorithms discussed so far can be described as **undirected**. This means that they "blindly" work their way out from the start state until the goal state is encountered. Intuitively, it seems that we should be able to search more quickly by preferring to expand nodes that are likely to be closer to the goal: If we know that the goal is to the east of the robot, then surely we should start the search by exploring actions that move the robot in that direction.

This intuition can be formalized through the idea of a **heuristic function** $h(s)$. A heuristic function maps from a state to the estimated cost of reaching the goal from that state. States with lower heuristic values are believed to be closer to the goal, and thus should be explored before states with higher heuristic values.

We can easily update Dijkstra's algorithm to take advantage this idea. The only change required is that the the Priority Queue representing the frontier will now be ordered by $f(s) = c(s) + h(s)$ where $c(s)$ represents the cost of reaching state s, and $h(s)$ represents the estimated cost of reaching the goal from s. The sum represents an estimate of the overall cost of the shortest path that passes through state s.

The resulting Algorithm is called A*.

The A* algorithm is guaranteed to find an optimal path as long as the heuristic function satisfies two requirements: First, it must never overestimate the true cost of reaching the goal. An overestimate could prevent a node from being expanded, even if that node is on the shortest path to the goal. Second, the heuristic function must be **consistent**. Formally, this means that $h(s) \leq h(s') + cost(s, s')$ for all states s and s' where $cost(s, s')$ represents the cost of the edge between s and s'. Informally, this means that the heuristic function should respect the actual step costs. When stepping from s to s', the heuristic function shouldn't drop by more than the cost of the step.

The "*" in A* was chosen to indicate that this is, in a sense, the last word in heuristic-based graph search algorithms. Not only is A* guaranteed to find a shortest path, it is also provably optimal in terms of the number of nodes expanded. Any algorithm that is guaranteed to find a shortest path must expand at least as many nodes as A* when using the same heuristic function.

Taking full advantage of A* requires the selection of a good heuristic function. A good heuristic function should be as close as possible to the true cost of reaching the goal, while being fast to compute and satisfying the consistency requirement. In general, it can be challenging to satisfy these requirements. However, in robotic path planning problems where the objective is to find a minimum length path, the straight-line distance to the goal is often a reasonable choice. The straight-line distance is easy to compute. It may underestimate the true cost, (because it doesn't take obstacles into account) but it can never overestimate: the straight-line distance always represents the shortest possible path to the goal. If the step cost is taken to be the distance traveled, then the straight line distance is consistent because no step can reduce the distance to the goal by more than the distance moved.

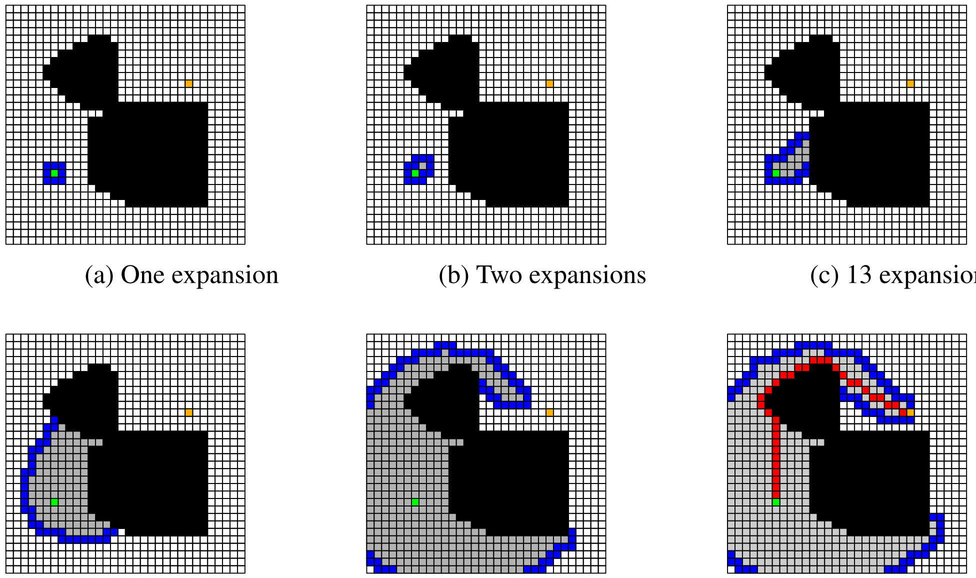
Figure 3.10 illustrates an A* search using the straight line heuristic. Notice that, while A* does not return the same path that was discovered by Dijkstra's algorithm, the two paths have the same length and both are optimal.

Stop and Think

3.13 The **Manhattan distance** is a count of the minimum number of edges between two states in a four-connected grid. Is Manhattan distance an admissible heuristic for the search problem in Question 3.9? ■

3.14 Repeat Question 3.9 using an A* search. Use the Manhattan distance to the goal as the heuristic function. For this problem, the fourth value in the Frontier tuples will represent $f(s) = c(s) + h(s)$. The first three expansions are done for you.

Expansions	Chosen	Frontier	Closed Set
0	–	$\langle g, -, 0, 2\rangle$	–
1	g	$\langle d, g, 6, 7\rangle$, $\langle h, g, 1, 4\rangle$	$\{g\}$
3	h	$\langle d, g, 6, 7\rangle$, $\langle e, h, 3, 5\rangle$, $\langle i, h, 2, 6\rangle$	$\{g, h\}$
3			
4			
5			



(a) One expansion (b) Two expansions (c) 13 expansions

(d) 100 expansions (e) 350 expansions (f) 373 expansion

Figure 3.10: A* search example.

3.15 In comparing Figure 3.10 to Figure 3.9 we can see that A* does appear to be "smarter". It tends to expand nodes that are closer to the goal before nodes that are farther away. On the other hand A* does expand some nodes that are in the "wrong" direction. For example, by expansion 100 it has expanded several nodes below and to the left of the start state, even those they are in the opposite direction from the goal. What's going on? Why doesn't A* always expand the node that is closest to the goal?

3.16 Imagine we want to encourage A* to avoid taking steps that move the robot close to an obstacle. We could attempt to accomplish this by adding 5 to the heuristic function for every state that is adjacent to an obstacle. Is it possible for the resulting heuristic to overestimate the cost to goal? Is the resulting heuristic consistent? Can you think of a better approach achieving the desired outcome?

3.17 Consider the heuristic function $h(s) = 0$ for all states s. Does this heuristic ever over-estimate the true cost? Is it consistent? Is it a useful heuristic? Why or why not?

3.3.4 The Efficiency of Grid-Based Search

The worst-case time complexity of both Dijkstra's algorithm and A* is usually described as $O(d^b)$ where d is the number of steps to the goal and b is the **branching factor** – the average number of

Dimensions (d)	Branching Factor (b)	Number of States (N)
1	$3^1 - 1 = 2$	$100^1 = 100$
2	$3^2 - 1 = 8$	$100^2 = 10{,}000$
3	$3^3 - 1 = 26$	$100^3 = 1{,}000{,}000$
4	$3^4 - 1 = 80$	$100^4 = 100{,}000{,}000$
5	$3^5 - 1 = 242$	$100^5 = 10{,}000{,}000{,}000$
6	$3^6 - 1 = 728$	$100^6 = 1{,}000{,}000{,}000{,}000$

Table 3.1: Branching factor and state space size as a function of dimensionality

successors for each state. This is a valid upper-bound, but it can give a dramatic over-estimate for problems where the pruning allowed by closed the set allows us to avoid considering many alternative paths that pass through the same state. When using a closed set, the total number of expansions is, at most, equal to the number of states N, since each state can be expanded at most once. Therefore, for problems with a finite number of state, a tighter bound is $O(Nb\ \log(N)))$:

Neighbors per expansion

Maximum expansions

Priority Queue and Set operations

$$\boxed{N}\ \boxed{b}\ \boxed{\log(N)}$$

This means that both Dijkstra's algorithm and A* are actually polynomial-time algorithms as a function of the number of states. The bad news is that, for grid-based discretizations of a configuration space, the number of states and the branching factor both grow exponentially with the number of dimensions in the problem.

Table 3.1 illustrates a grid-based discretization where each dimension is divided into 100 equal increments and each grid cell is connected to its immediate neighbors. The take-home message from this table is that grid-based search works well for two-dimensional search problems, but quickly becomes impractical as the number of dimensions increases. This is a significant issue, given that practical search problems can easily have six dimensions or more. The remainder of this chapter will consider alternatives to grid-based search that are suitable for high-dimensional path planning.

3.4 Sampling-Based Search

Sampling-based search algorithms abandon the completeness and optimally guarantees of grid-based search in exchange for better performance in high-dimensional configuration spaces. Instead of creating a dense discretization, sampling based algorithm create a sparse web of nodes by placing edges between randomly generated configurations.

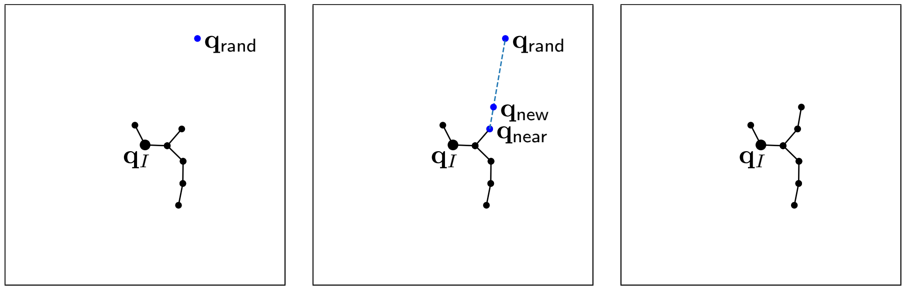


Figure 3.11: One iteration of the rapidly exploring random tree algorithm. Left: a random configuration $\mathbf{q}_{\mathrm{rand}}$ is selected. Center: the nearest node in the existing tree $\mathbf{q}_{\mathrm{near}}$ is selected for expansion. A new node $\mathbf{q}_{\mathrm{new}}$ is created by running a local planner to find an action that moves in the direction of the random node. Right: The new node is added to the tree.

3.4.1 Rapidly Exploring Random Trees

The Rapidly Exploring Random Tree (RRT) algorithm is an undirected search algorithm that incrementally builds a tree outward from the starting configuration until the goal configuration is reached. Figure 3.15 illustrates one iteration of RRT. First, a configuration $\mathbf{q}_{\mathrm{rand}}$ is sampled from the full configuration space. Next, we select the node in the tree $\mathbf{q}_{\mathrm{near}}$ that is nearest to the randomly generated point for expansion. Finally, we add a node $\mathbf{q}_{\mathrm{new}}$ to the tree by taking a step from $\mathbf{q}_{\mathrm{near}}$ in the direction of $\mathbf{q}_{\mathrm{rand}}$. The full algorithm is outlined in Listing 4.

The basic algorithm presented in Listing 4 only handles the construction of the tree. The algorithm can be augmented to return a path by adding a check to see if newly created nodes are close enough to the goal configuration to allow a connection.

```python
def rrt(problem, q_init, tree_size):
    """ Build a rapidly exploring random tree.

    Args:
        problem:   a problem instance that provides three methods:

            problem.random_state() -
               Return a randomly generated configuration
            problem.select_input(q_rand, q_near) -
               Return an action that would move the robot from
               q_near in the direction of q_rand
            problem.new_state(q_near, u) -
               Return the state that would result from taking
               action u in state q_near

        q_init:     the initial state
        tree_size:  the number of nodes to add to the tree

    Returns:
        A tree of configurations rooted at q_init

    """
    tree = Tree(q_init)  # Make the start state the root of the tree
    while tree.num_nodes() < tree_size:
        q_rand = problem.random_state()
        q_near = nearest_neighbor(tree, q_rand)
        u = problem.select_input(q_rand, q_near)
        q_new = problem.new_state(q_near, u)
        tree.add_node(q_new):
        tree.add_edge(q_near, q_new)
    return tree
```

Listing 4: The Rapidly Exploring Random Tree (RRT) algorithm.

The strength of the RRT algorithm lies in its tendency to quickly move away from the initial configuration into unexplored areas of the configuration space. The Voronoi diagram in Figure 3.12 provides an intuition for this behavior. A Voronoi diagram partitions space into regions based on the nearest neighbor boundaries. As can be seen in Figure 3.12, nodes at the edge of the tree have much larger Voronoi regions than nodes on the interior. This means that a randomly generated configuration is more likely to land within the Voronoi region of one of these edge nodes. This Voronoi bias tends to pull the tree into unexplored areas of the configuration space.

Figure 3.13 illustrates the progress of the RRT algorithm on the triangle robot example. Notice that the final path is clearly not optimal. In fact the RRT algorithm is neither optimal nor complete. It is, however, **probabilistically complete**: if a path to the goal exists, the probability of finding a path approaches one as the number of iterations approaches infinity.

In practice, RRT planners typically perform some post-processing to smooth out paths once they have been discovered. This can be done, for example, by selecting non-neighboring points on the path and

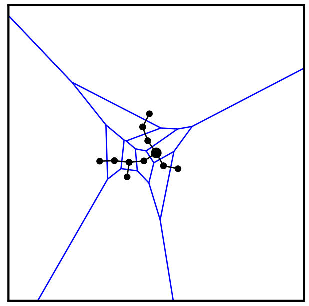


Figure 3.12: Voronoi diagram for a partially constructed random tree. The nodes at the edge of the tree have larger Voronoi regions, and so are more likely to be expanded.

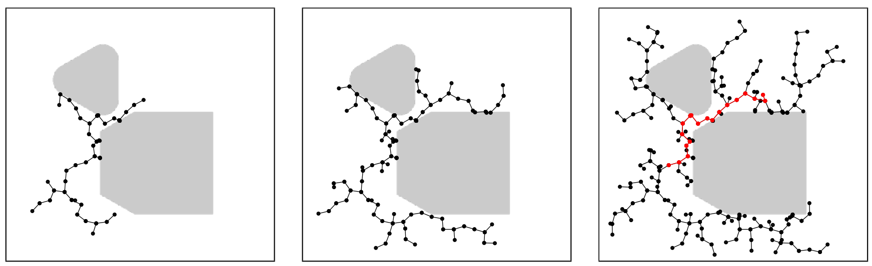


Figure 3.13: Example of applying the RRT algorithm to the triangle robot navigation problem from Figure 3.4. The final path is shown in red in the right-most figure.

attempting to find direct connections between them.

RRT's for Non-Holonomic Path Planning

The Rapidly Exploring Random Tree algorithm is well suited to non-holonomic path-planning problems. For the holonomic example in Figure 3.13, `select_input` simply creates a short line segment from $\mathbf{q}_{\text{near}}$ in the direction of $\mathbf{q}_{\text{rand}}$. In the case of non-holonomic planning, `new_state` and `select_input` must be implemented to respect the non-holonomic constraints.

Figure 3.14 shows an example of RRT path planning for a non-holonomic wheeled vehicle that may only move forward and has a maximum turning radius. In the initial configuration, the robot is located inside a narrow passage and is pointed away from the goal location. The RRT algorithm works outward from the initial configuration to the goal in steps that respect the non-holonomic constraints.

Stop and Think

3.18 Take a moment to examine the RRT algorithm in Listing 4. Which steps in the algorithm do you expect to be most computationally expensive? Why? ■

3.19 The basic RRT algorithm in Listing 4 is completely undirected. A common optimization is to bias search in the direction of the goal by selecting the goal configuration as $\mathbf{q}_{\text{rand}}$ with some small

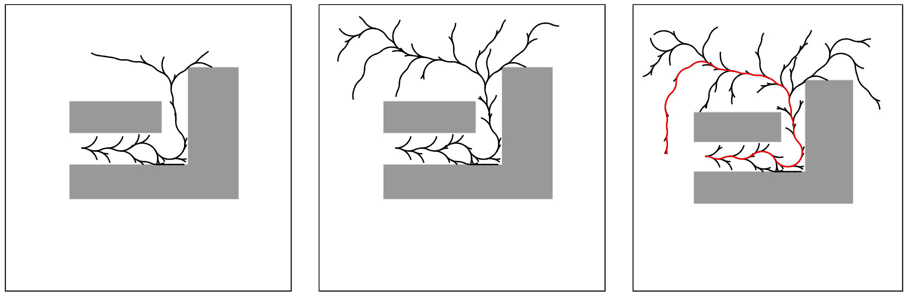


Figure 3.14: Example of applying the RRT algorithm to a non-holonomic path-planning problem.

probability. Why not select the goal configuration 100% of the time?

3.4.2 Probabilistic Roadmaps

One weakness of the RRT algorithm is that it must explore the configuration space from scratch for each search. In domains that involve repeated searches within the same environment, it may be more efficient to pre-compute a graph that maps the connectivity of $\mathcal{C}_{free}$. The Probabilistic Road Map (PRM) algorithm takes this approach. The algorithm proceeds by generating random configurations from $\mathcal{C}_{free}$ and attempting to connect those configurations to all existing configurations within some fixed radius. Listing 4 provides the complete algorithm and Figure 3.15 illustrates one iteration.

```python
def prm(problem, delta, roadmap_size):
    """ Create a Probabilistic Roadmap.

    Args:
        problem:  a problem instance that provides two methods:

           problem.random_state() -
               Return a random state from C_free
           problem.no_collision(q1, q2) -
               Return True if there is a collision free path
               from q1 to q2

        delta: Distance threshold for connecting neighboring states
        roadmap_size: Number of nodes in the completed Roadmap

    Returns:
        A graph representing a Probabilistic Roadmap

    """
    graph = Graph()
    while graph.num_nodes() < roadmap_size:
        q_rand = problem.random_state()
        graph.add_node(q_rand)
        for q in neighbors(graph, q_rand, delta):
            if problem.no_collision(q, q_rand):
                graph.add_edge(q, q_rand)
    return graph
```

Listing 5: The Probabilistic Road Map Algorithm (PRM).

Once a roadmap has been created it can be used for path-planning by adding the start and goal configurations to the graph and using any of the graph search algorithms from Section 3.3.

The PRM algorithm is not well suited to non-holonomic path planning problems. The PRM graph must be undirected to facilitate planning from an initially unknown start configuration to an unknown goal configuration. As with the wheeled vehicle example from Figure 3.14, non-holonomic domains often do not allow for “reversible” actions of the type that can be represented in an undirected graph.

Stop and Think

3.20 What is the difference between implementing `problem.random_state` for use with the PRM algorithm vs. RRT? ■

3.21 The original formulation of the PRM algorithm only added edges between nodes that were not already in the same connected component. How would this impact the probability of finding a path in the resulting map? How would this impact the quality of the discovered paths? ■

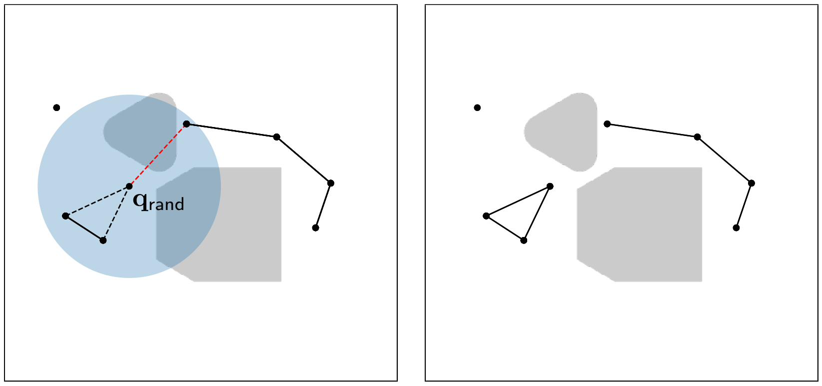


Figure 3.15: One iteration of the probabilistic Roadmap algorithm. The figure on the left shows the randomly generated configuration $\mathbf{q}_{\text{rand}}$ connected by dotted lines to the existing points that are within range. The dotted red line will not be added to the roadmap because it passes through an obstacle. The figure on the right shows the roadmap after $\mathbf{q}_{\text{rand}}$ and the appropriate edges have been added.

3.5 References and Further Reading

Discrete-space planning algorithms are a foundational idea in artificial intelligence and are covered in AI textbooks including (Russell & Norvig, 2010) and (Poole & Mackworth, 2017). The A* algorithm was invented at SRI in the 1960s for use with Shakey the robot. The Shakey project was an ambitious early effort in autonomous robotics that resulted in key innovations in planning, navigation, and computer vision. Nils J. Nilsson recounts the history of that project, along with a broader history of early research in artificial intelligence in (Nilsson, 2009). The A* algorithm was formally introduced, and proven to be optimal, in (Hart et al., 1968).

The use of configuration spaces as a formalism in planning was introduced in (Lozano-Perez, 1983). Steven LaValle's *Planning Algorithms* textbook provides a thorough introduction to configuration spaces and provides in-depth descriptions of each of the planning algorithms discussed in this chapter. (LaValle, 2006).

The rapidly exploring random tree algorithm was invented by Steven LaValle (LaValle, 1998). Many variations have been published that improve or modify some aspect of the algorithm's behavior. One notable example is the RRT* algorithm, which dynamically "re-wires" the search tree in order to make local improvements when new nodes are added (Karaman & Frazzoli, 2010). While the original RRT algorithm provides no guarantees about the quality of the final solution, it can be shown that the RRT* algorithm converges on the lowest-cost path as the number of samples approaches infinity.

The probabilistic roadmap algorithm was introduced in (Kavraki et al., 1996). As with RRT, PRM has served as the basis for many variations in the planning literature.

The Open Motion Planning Library (OMPL) https://ompl.kavrakilab.org/ is an open source library that includes high-quality implementations of sampling based planning algorithms including RRT, RRT*, PRM and many others.