CS228 - Relations

Nathan Sprague

March 5, 2014

Material in these slides is from “Discrete Mathematics and Its Applications 7e”, Kenneth Rosen, 2012.

Relations

Definition

Let A and B be sets. A *binary relation from A to B* is a subset of $A \times B$.

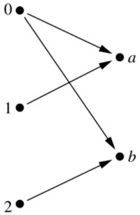
- aRb denotes that $(a, b) \in R$
- $a \not R b$ denotes that $(a, b) \notin R$

Example

- A = All US city names.
  B = All US states.
  $R = \{(a, b) \mid \text{A city with name } a \text{ is located in state } b.\}$
  - (Harrisonburg, Virginia) $\in R$
  - Harrisonburg R Virginia
- Relations are not functions:
  - Franklin R Virginia
  - Franklin R Ohio

Displaying Relations

- $A = \{0, 1, 2\}$
  $B = \{a, b\}$
  $R = \{(0, a), (0, b), (1, a), (2, b)\}$



R	a	b
0	×	×
1	×	
2		×

Relations On a Set

Definition

A *relation on a set A* is a relation from A to A.

Relations on the set of integers:

- $R_1 = \{(a, b) \mid a \leq b\}$
- $R_2 = \{(a, b) \mid a > b\}$
- $R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$

Which relations contain $(1, 1), (1, 2), (2, 1), (1, -1)$?

Refelexive Relations

Definition

A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

Relations on the set of integers:

- $R_1 = \{(a, b) \mid a \leq b\}$
- $R_2 = \{(a, b) \mid a > b\}$
- $R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$

Which are reflexive?

Symmetric and Antisymmetric Relations

Definition

A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.

A relation R on a set A such that for all $a, b \in A$, if $(b, a) \in R$ and $(a, b) \in R$, then $a = b$ is called *antisymmetric*.

Relations on the set of integers:

- $R_1 = \{(a, b) \mid a \leq b\}$
- $R_2 = \{(a, b) \mid a > b\}$
- $R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$

Which are symmetric? antisymmetric?

Transitive Relations

Definition

A relation R on a set A is called *transitive* if whenever $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$, for all $a, b, c \in A$.

Relations on the set of integers:

- $R_1 = \{(a, b) \mid a \leq b\}$
- $R_2 = \{(a, b) \mid a > b\}$
- $R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$

Which are transitive?

Composites of Relations

Definition

Let R be a relation from set A to set B and S a relation from B to set C. The *composite of* R *and* S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

Example

- A = All first names in the US.
  B = All city names in the US.
  C = All US states.
  $R = \{(a, b) \mid \text{A person with name } a \text{ is located in city } b.\}$
  $S = \{(b, c) \mid \text{A city with name } b \text{ is located in state } c.\}$
- What is $S \circ R$?

Example

- A = All first names in the US.
  B = All city names in the US.
  C = All US states.
  $R = \{(a, b) \mid$ A person with name a is located in city $b.\}$
  $S = \{(b, c) \mid$ A city with name b is located in state $c.\}$
- What is $S \circ R$?
- $S \circ R = \{(a, c) \mid$ A person with name a is located in state $c.\}$

Powers of Relations

Definition

Let R be a relation on the set A. The powers R^n, $n = 1, 2, 3, \ldots$, are defined recursively by

$R^1 = R$ and $R^{n+1} = R^n \circ R$.