CS228 - Probability

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Material in these slides is from "Discrete Mathematics and Its Applications 7e", Kenneth Rosen, 2012.

Probability

Terminology:

- An **experiment** is a procedure that yields on of a given set of outcomes.
- The **sample space** is the set of possible outcomes.
- An **event** is a subset of the sample space

Definition of the probability of an event:

If S is a finite sample space of equally likely outcomes, and E is an event (a subset of S), then the **probability** of E is

$$p(E) = \frac{|E|}{|S|}.$$

Examples

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Solution: By the product rule there are $6^2 = 36$ possible outcomes. Six of these sum to 7. Hence, the probability of obtaining a 7 is $6/36 = 1/6$.

The Probability of Complements

Theorem

Let E be an event in sample space S. The probability of the event $\overline{E} = S - E$, the complementary event of E, is given by

$$p(\overline{E}) = 1 - p(E).$$

Example

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Solution: Let E be the event that at least one of the 10 bits is 0. Then $\overline{E}$ is the event that all of the bits are 1s. The size of the sample space S is 2^{10}. Hence,

$$p(E) = 1 - p(\overline{E}) = 1 - \frac{|\overline{E}|}{|S|} = 1 - \frac{1}{2^{10}} = 1 - \frac{1}{1024} = \frac{1023}{1024}.$$

The Probability of Unions

Theorem

Let E_1 and E_2 be events in the sample space S. Then

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2).$$

(Follows from the inclusion-exclusion principle.)

Example

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Solution: Let E_1 be the event that the integer is divisible by 2 and E_2 be the event that it is divisible 5. Then the event that the integer is divisible by 2 or 5 is $E_1 \cup E_2$ and $E_1 \cap E_2$ is the event that it is divisible by 2 and 5. It follows that:

$$\begin{aligned} p(E_1 \cup E_2) &= p(E_1) + p(E_2) - p(E_1 \cap E_2) \\ &= 50/100 + 20/100 - 10/100 = 3/5. \end{aligned}$$

Probabilities

Previous definition assumes that all outcomes are equally likely. There is a more general definition of probabilities that avoids this restriction.

- Let S be the sample space of an experiment with a finite number of outcomes. We assign a probability $p(s)$ to each outcome s, so that:
  - $0 \leq p(s) \leq 1$ for each $s \in S$
  - $\sum_{s \in S} p(s) = 1$
- The function p is called a probability distribution.

Probability of An Event

Definition

The probability of the event E is the sum of the probabilities of the outcomes in E:

$$p(E) = \sum_{s \in E} p(s)$$

Example

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Solution: We want the probability of the event $E = \{1, 3, 5\}$. We have $p(3) = 2/7$ and $p(1) = p(2) = p(4) = p(5) = p(6) = 1/7$. Hence,

$$
\begin{aligned}
p(E) &= p(1) + p(3) + p(5) \\
&= 1/7 + 2/7 + 1/7 = 4/7.
\end{aligned}
$$

Independence

Definition

The events E and F are independent if and only if:

$$p(E \cap F) = p(E)p(F).$$

Example: Showing Independence

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Solution: There are eight bit strings of length four that begin with a 1, and eight bit strings of length four that contain an even number of 1s.

- Since the number of bit strings of length 4 is 16, $p(E) = p(F) = 8/16 = 1/2$
- Since $E \cap F = \{1111, 1100, 1010, 1001\}$, $p(E \cap F) = 4/16 = 1/4$
- E and F are independent because $1/2 \cdot 1/2 = 1/4$.

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Solution: We assume that the dice rolls are mutually independent. The propability of rolling 5 or above on a single dice is $1/3$, therefore the probabilty of rolling 5 or above on three dice is $(1/3)^3 = 1/9$.