CS228 - Induction

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Material in these slides is from "Discrete Mathematics and Its Applications 7e", Kenneth Rosen, 2012.

Principle of Mathematical Induction

To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

- Basis Step: We verify that $P(1)$ is true.
- Inductive Step: We show that the conditional statement $P(k) \rightarrow P(k+1)$ is true for all positive integers k.

Principle of Mathematical Induction (Generalized)

To prove that $P(n)$ is true for all integers $n \geq b$ for a fixed integer b, where $P(n)$ is a propositional function, we complete two steps:

- Basis Step: We verify that $P(b)$ is true.
- Inductive Step: We show that the conditional statement $P(k) \rightarrow P(k+1)$ is true for all integers k with $k >= b$.

Example (1/2)

Show that if n is a positive integer, then $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

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Basis Step: P(1) is true, because $1 = \frac{1(1+1)}{2}$

Inductive Step: We assume that $P(k)$ is true for an arbitrary positive integer k. In other words, we assume that $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$. We must show that $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$ is true under this assumption.

Example (2/2)

Inductive Step (Continued):

$$\sum_{i=1}^{k} i \overset{IH}{=} \frac{k(k+1)}{2}$$

Add $k+1$ to both sides:

$$\sum_{i=1}^{k+1} i = \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

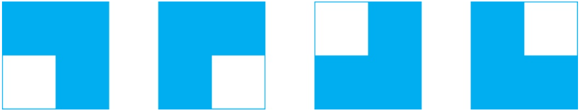
$$= \frac{(k+1)(k+2)}{2}$$ This completes the inductive step.

Tiling Example (1/2)

Show that every $2^n \times 2^n$ sized checkerboard with one square removed can be tiled using right trionimoes.

Inductive Proof: Let $P(n)$ be the proposition that every $2^n \times 2^n$ sized checkerboard with one square removed can be tiled using right trionimoes.

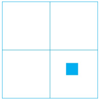
Basis Step: $P(1)$ is true because each of the four 2×2 checkerboards with one square removed can be tiled using one trionimo:

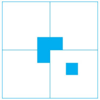


Tiling Example (2/2)

Inductive Step: Assume that $P(k)$ is true for every $2^k \times 2^k$ checkerboard, for some positive integer k.

Consider a $2^{k+1} \times 2^{k+1}$ checkerboard with one square removed. Split this checkerboard into four checkerboards of size $2^k \times 2^k$, by dividing it in half in both directions.





Remove a square from one of the four$2^k \times 2^k$ checkerboards. By the inductive hypothesis, this board can be tiled. Also by the inductive hypothesis, the other three boards can be tiled with the square from the corner of the center of the original board removed. We can then cover the three adjacent squares with a trionimo.

Inductive Proof Template (p. 329)

1. Express the statement that is to be proved in the form "for all $n \geq b, P(n)$" for a fixed integer b.
2. Write out the words "Basis Step." Then show that $P(b)$ is true.
3. Write out the words "Inductive Step."
4. State, and clearly identify the inductive hypothesis, in the form "Assume that $P(k)$ is true for an arbitrary fixed integer $k \geq b$."
5. State what needs to be proved under the assumption that the inductive hypothesis is true. Write out $P(k+1)$.
6. Prove the statement $P(k+1)$ making use of the assumption $P(k)$.
7. Clearly identify the conclusion of the inductive step.
8. State the conclusion: "By mathematical induction P(n) is true for all integers n with $n \geq b$."