CS228 - Closures (Section 9.4)

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March 19, 2014

Material in these slides is from "Discrete Mathematics and Its Applications 7e", Kenneth Rosen, 2012.

Closures

- "In general, let R be a relation on a set A. R may or may not have some property **P** such as reflexivity, symmetry or transitivity. If there is a relation S with property **P** containing R such that S is a subset of every relation with property **P** containing R, then S is called the **closure** of R with respect to **P**."
- Less formally: The **P**-closure of R is the smallest superset of R that has property **P**.

Reflexive Closure

- **Reflexive closure of** $R = R \cup \triangle$
  - where $\triangle = \{(a, a) \mid a \in A\}$ (diagonal relation).
- Example: $R = \{(a, b) \mid a < b\}$
  - $R \cup \triangle = \{(a, b) \mid a \leq b\}$

Reflexive closure of $R = \{(1, 1), (1, 2), (2, 1), (3, 2)\}$ on the set $S = \{1, 2, 3, 4\}$?

Symmetric Closure

- **Symmetric closure of** $R = R \cup R^{-1}$
  - $R^{-1} = \{(b, a) \mid (a, b) \in R\}$ inverse
  - $\bar{R} = \{(a, b) \mid (a, b) \notin R\}$ complement

Example: $R = \{(a, b) \mid a > b\}$
$R \cup R^{-1} =$

Symmetric Closure

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Example: $R = \{(a, b) \mid a > b\}$

$R \cup R^{-1} = \{(a, b) \mid a \neq b\}$

Transitive Closure

Not so simple...

$R = \{(1,3),(1,4),(2,1),(3,2)\}$

Can't just add (1,2), (2,3), (2,4), (3,1) missing e.g., (3,4)

Paths

- *Graph* G is set of V vertices and E edges (edge is an ordered pair (a, b) of vertices)
- *Path* from a to b in G is a sequence of edges
  - Denoted by $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ (length n) where $x_0 = a$ and $x_n = b$.
- A *circuit* or *cycle* is path of length $n \geq 1$ that begins and ends at same vertex.

Transitive Closure

Theorem (page 600):

There is a path of length n from a to b iff $(a, b) \in R^n$

Recall: $R^1 = R$ and $R^{n+1} = R^n \circ R$

Proof by induction:

true when $n = 1$ (by definition)

assume true for k (inductive hypothesis)

path of length $k + 1$ from (a, b) iff there is some c s.t.:

$(a, c) \in R$ and $(c, b) \in R^k$

LHS: path of length 1 to c

RHS: path of length n from c to b

By IH, c exists iff $(a, b) \in R^{k+1}$

Connectivity Relation

Let R be a relation on set A. The *Connectivity relation* R^* consists of the pairs (a, b) such that there is a path of length at least one from a to b in R.

$$R^* = \bigcup_{n=1}^{\infty} R^n$$

(Note that $R^* = \bigcup_{n=1}^{\infty} R^n = \bigcup_{n=1}^{n} R^n$)

Connectivity Relation Example

- Example: $R = \{(a, b) \mid \text{state } a \text{ borders state } b\}$
  - What is R^1 for Virginia? (VA,WV), (VA,MD), (VA,NC), (VA,TN), (VA,KY)
  - What is R^2 for Virginia? (VA,VA), (VA,OH), (VA,PA), (VA,DE), ...
  - R^n = crossing exactly n state borders
  - R^* = crossing any # of borders (all pairs except with Alaska / Hawaii)

Transitive Closure

Theorem: Transitive closure of R equals R^*

Proof sketch:

1. R^* contains R by definition
2. R^* is transitive — just follow the paths
3. Any transitive S that contains R must also contain R^* (see page 601)

Computing Transitive Closure

Theorem: Let M_R be the zero-one matrix of R on a set with n elements

$$M_{R^*} = M_R \vee M_R^{[2]} \vee \cdots \vee M_R^{[n]}$$

Computing Transitive Closure

procedure TRANSITIVE CLOSURE($\mathbf{M}_R$: 0-1 $n \times n$ matrix)
  $\mathbf{A} := \mathbf{M}_R$
  $\mathbf{B} := \mathbf{A}$
  for $i := 2$ to n
    $\mathbf{A} := \mathbf{A} \odot \mathbf{M}_R$
    $\mathbf{B} := \mathbf{B} \vee \mathbf{A}$
  return $\mathbf{B}$

$O(n^4)$ because $n * n^3$ (for $\odot$)

Warshall's Algorithm

- Possible to calculate transitive closure in $O(n^3)$ steps
- Algorithm based on *interior vertices* of the paths
- Construct sequence of zero-one matrices $W_k = [w_{ij}^{(k)}]$
  - $w_{ij}^{(k)} = 1$ if there is a path from v_i to v_j such that all interior vertices are in the set $\{v_1, v_2, \ldots, v_k\}$
- Note that $W_n = M_{R^*}$

Warshall's Algorithm

procedure $\text{Warshall}(\mathbf{M}_R : \text{0-1 } n \times n \text{ matrix})$

  $\mathbf{W} := \mathbf{M}_R$

  for $k := 1$ to n

    for $i := 1$ to n

      for $j := 1$ to n

        $w_{ij} := w_{ij} \vee (w_{ik} \wedge w_{kj})$

  return $\mathbf{W}$

Warshall Example

$$\mathbf{W}_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Warshall Example

$$\mathbf{W}_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \mathbf{W}_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{W}_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \mathbf{W}_4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$