Geodesic Universal Molecules

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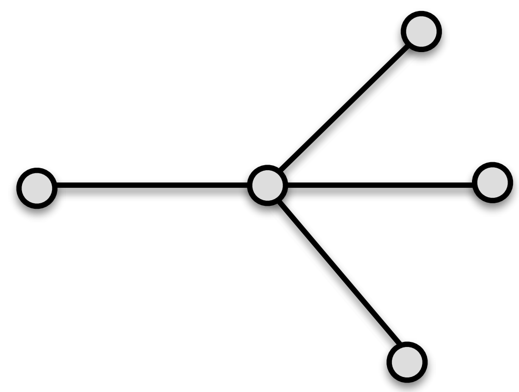
(joint work with Ileana Streinu)

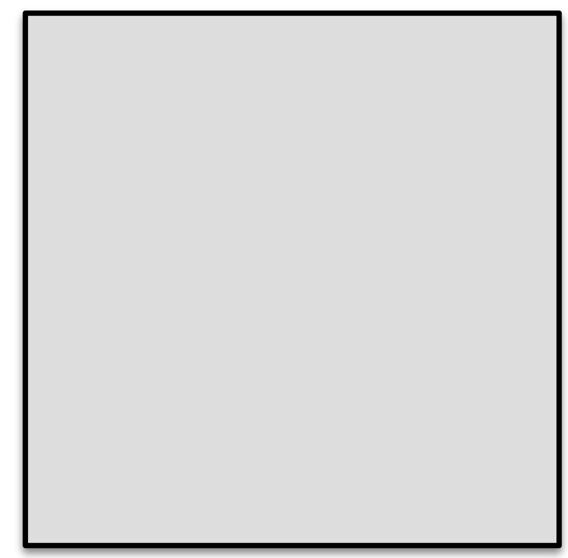
Tree Folding Problem

TreeMaker [Lang, 1996]

Given:

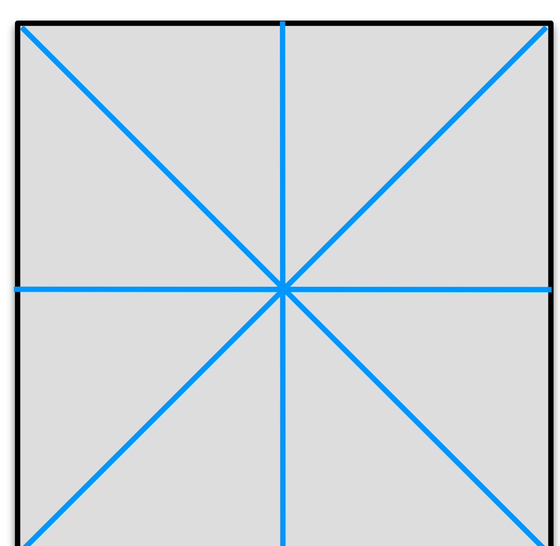


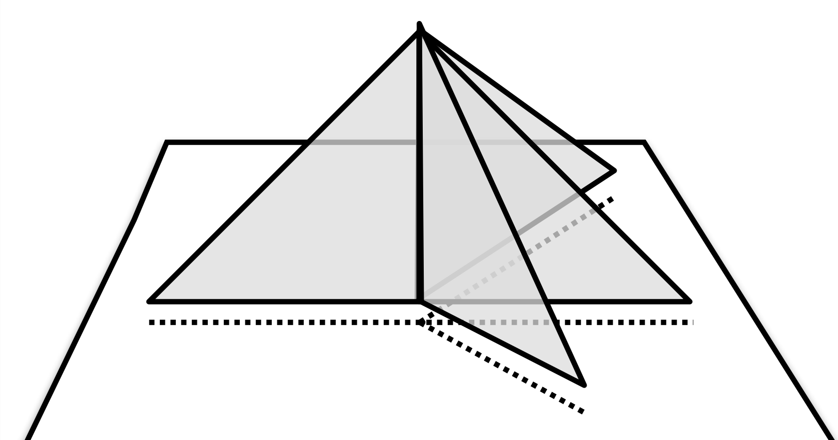
tree



“compatible” polygon

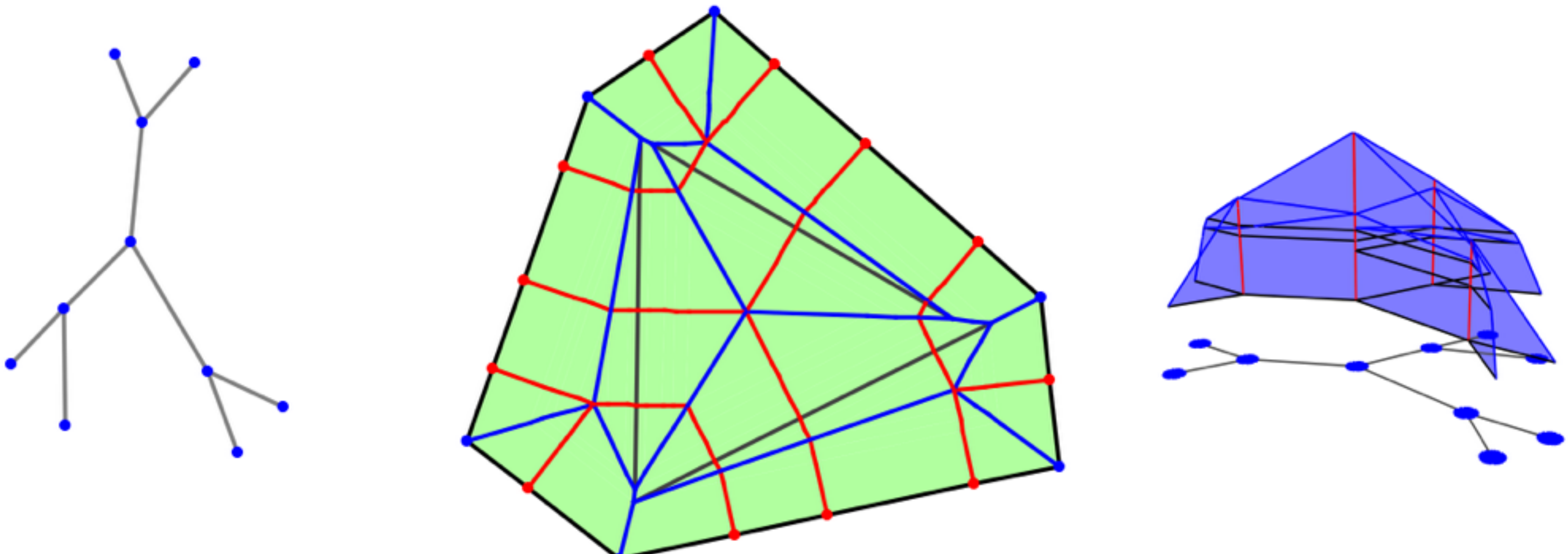
Goal:





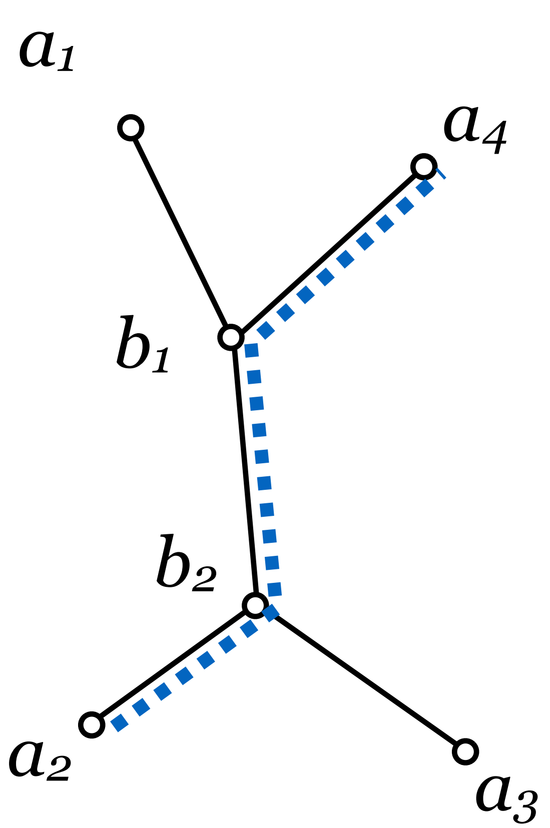
Universal Molecule

[Lang, 1996]

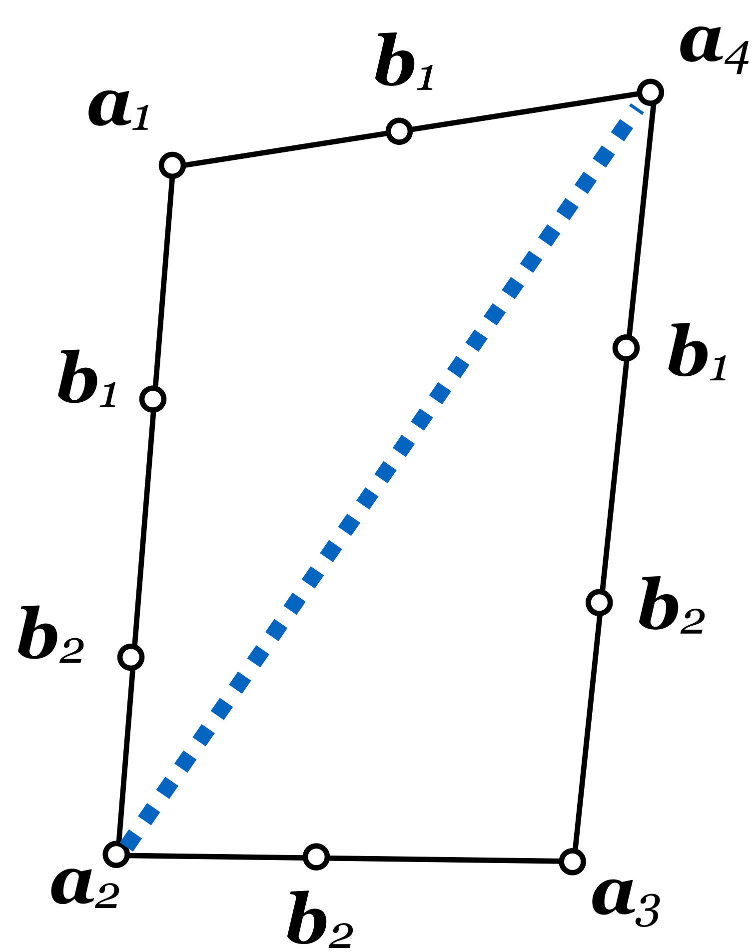


Brief Background

Universal Molecule Input: Tree & “Compatible” Polygon



positive-weight metric tree (with ordering)



Lang polygon

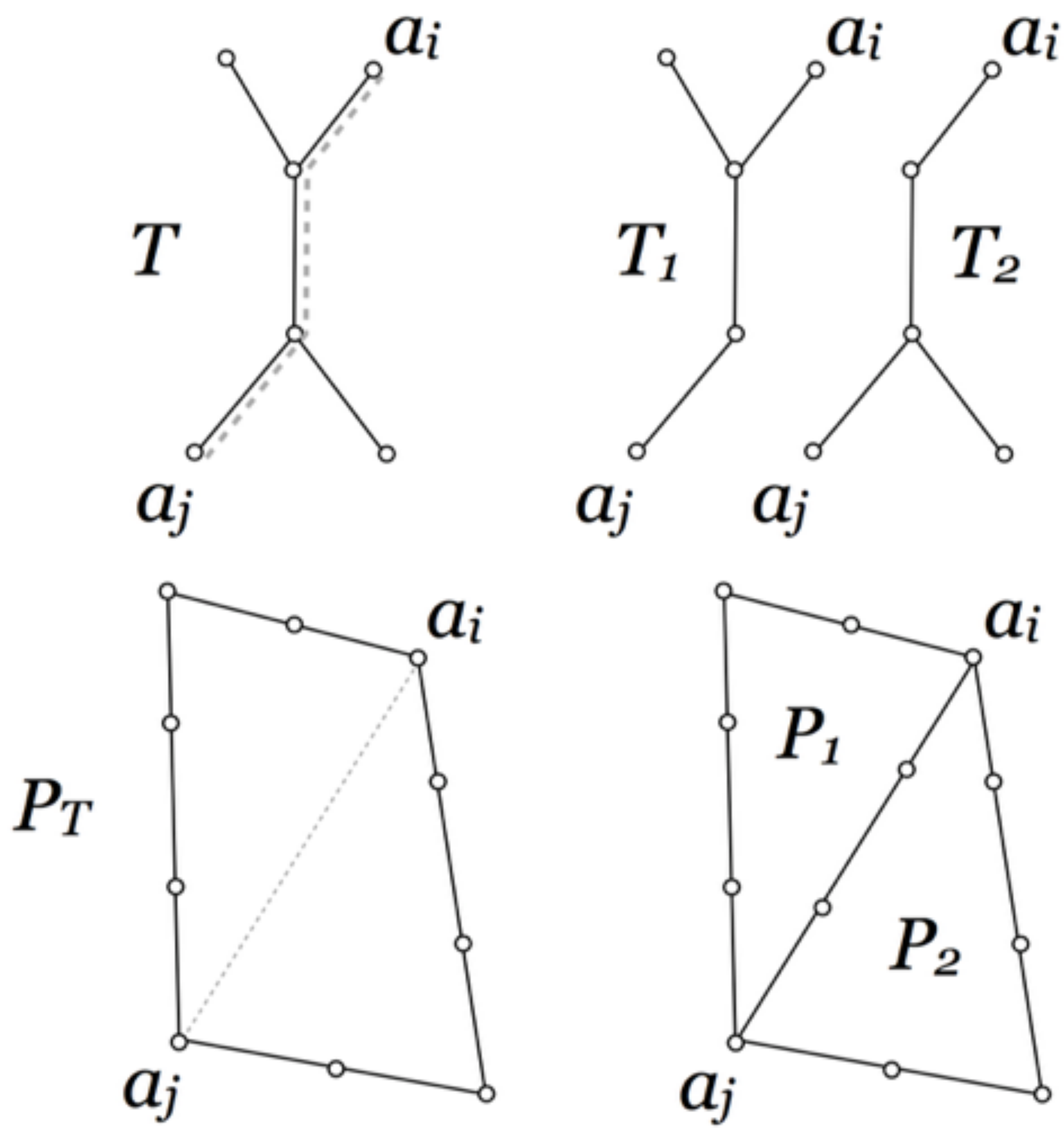
“Compatible” means:

1. Convex doubling polygon
2. Distance between two vertices is greater than or equal to corresponding distance in tree. (**Lang property**)

Implies:

- Vertices corresponding to internal nodes are straight.

Split Operation



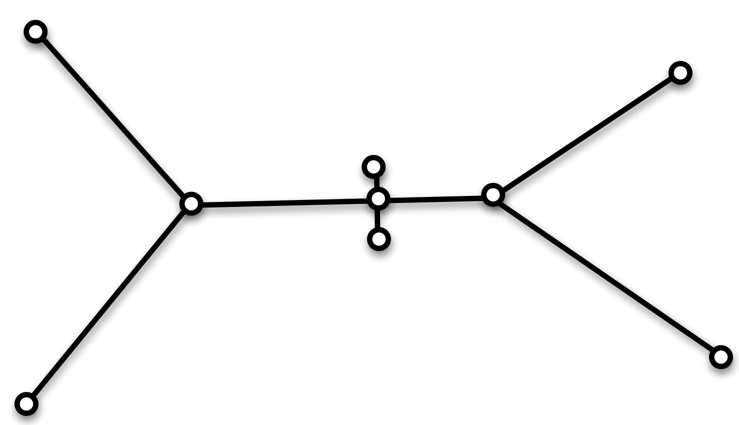
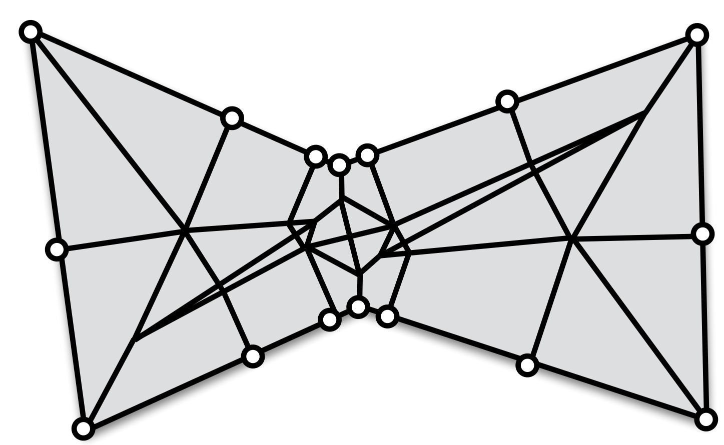
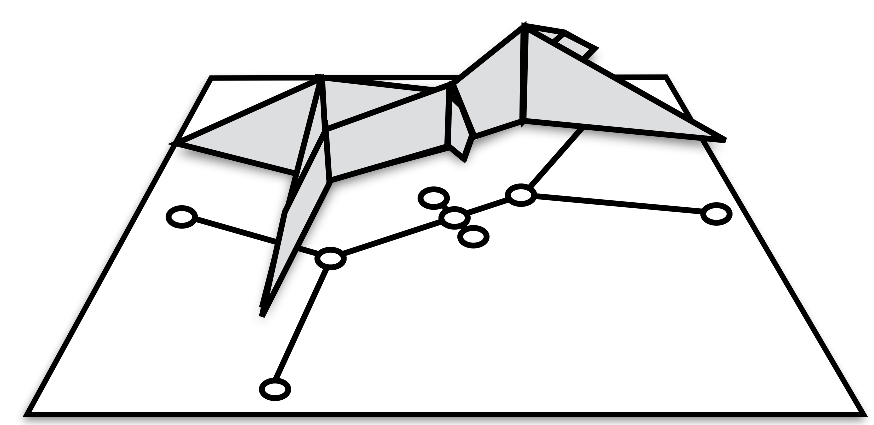
Universal Molecule Algorithm

- Better to see something live.

- **Main takeaway**: The **sweep** of a Lang polygon *(T, P_T)* continues in both tree and polygon until Lang property is violated, at which point both are split and the sweep continues independently in each.

Our Goal:

- Generalize the universal molecule to non-convex polygons:

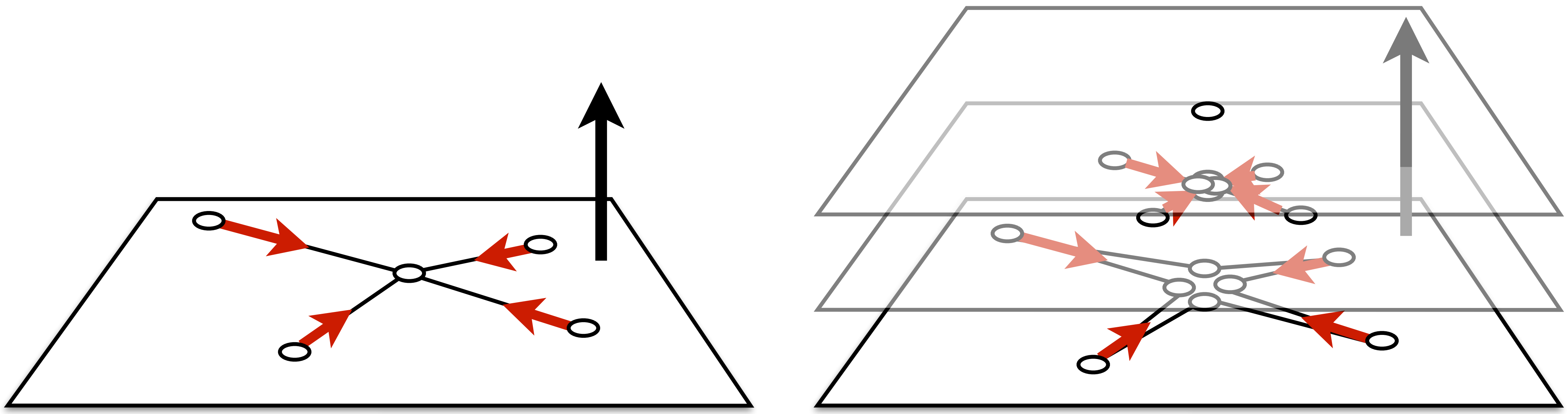
Key Ingredients

- Characterization of the folded surfaces that is independent of the algorithm (**Lang surfaces**).
- Generalization of Lang polygon and sweep to non-convex polygons (**Geodesic Lang polygons**).
- Description of **Geodesic Universal Molecule Algorithm**.

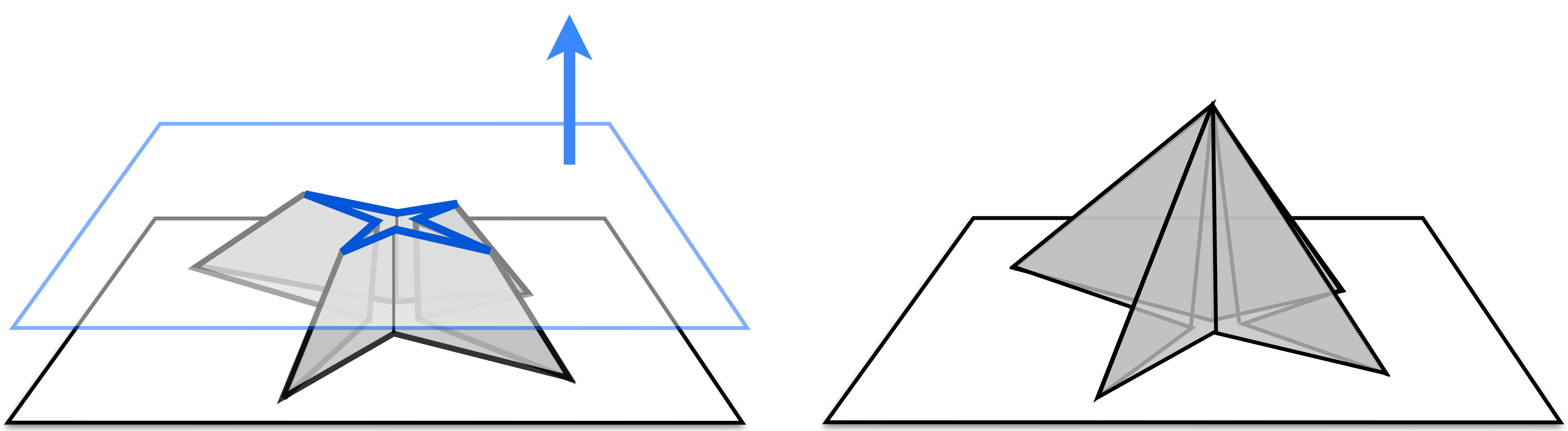
Lang Surfaces

- Inductive definition.
- Two building block surfaces.
- Two gluing operations.

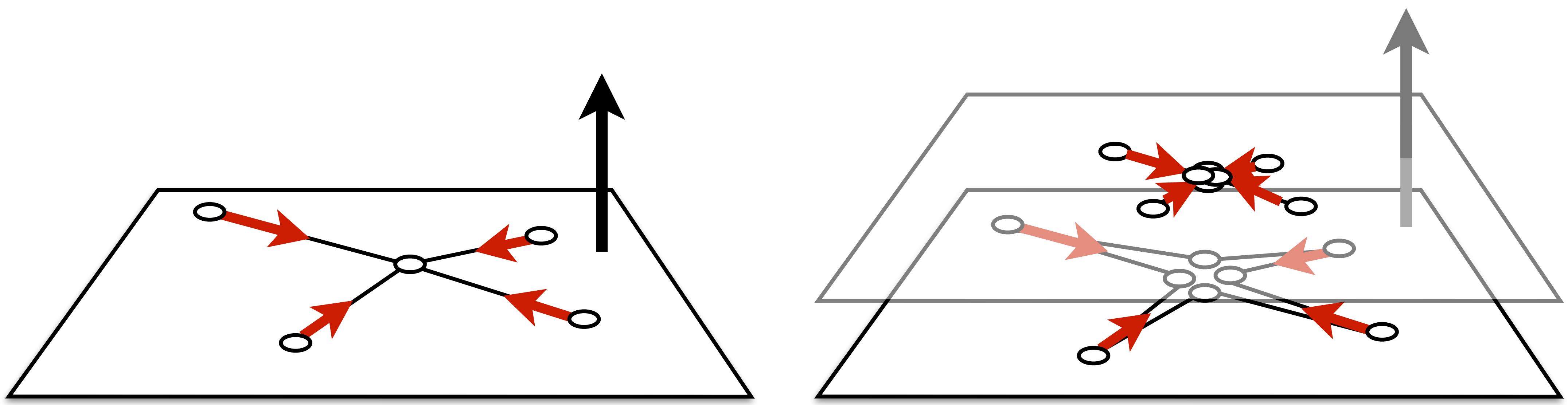
Extrusion Disk



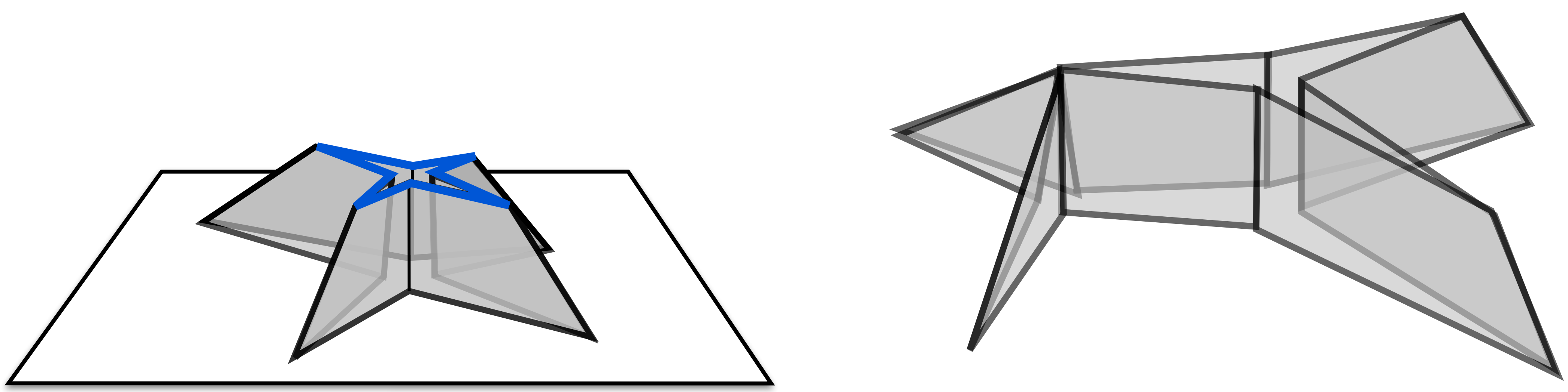
constraint: speeds chosen so
all arcs shrink simultaneously



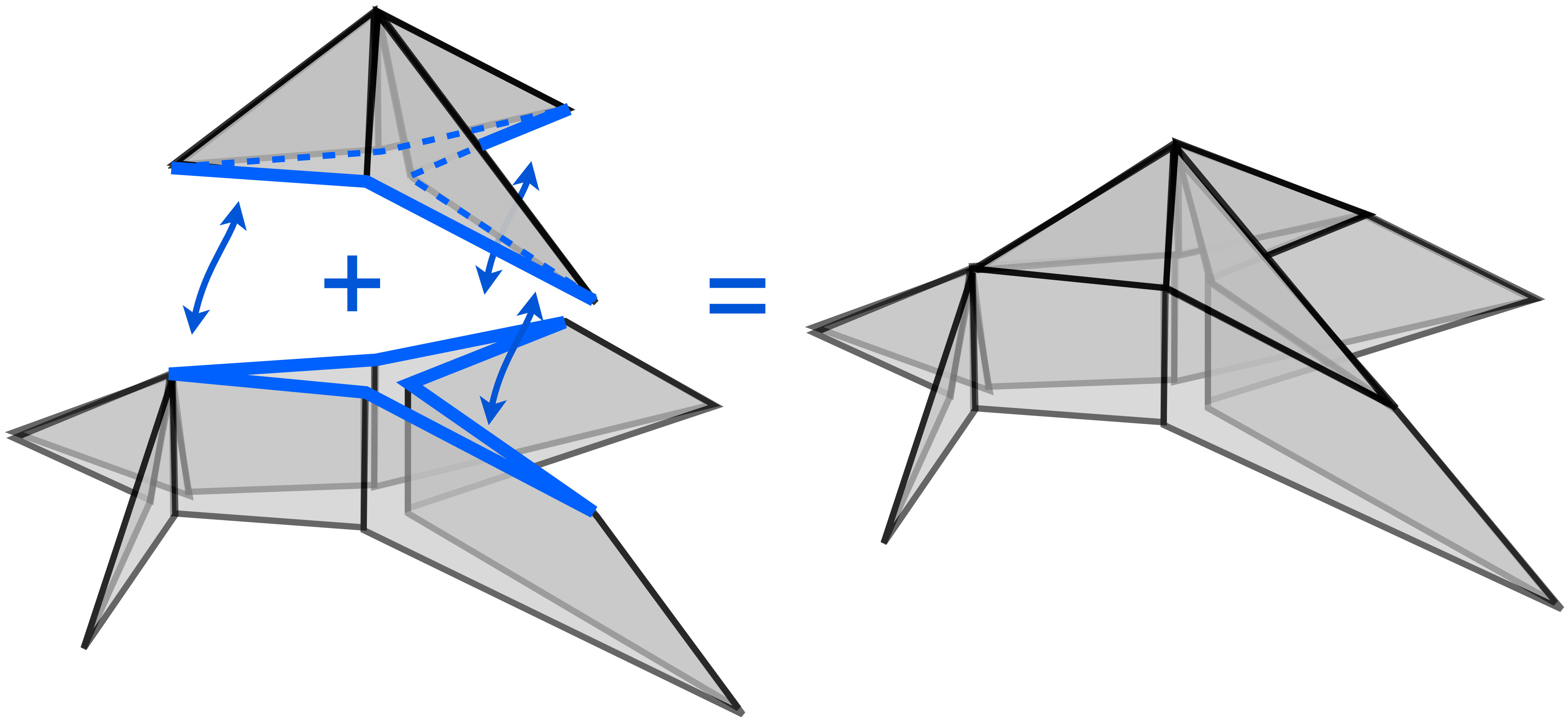
Extrusion Ring



no constraint on speeds
Note: can grow as well as shrink

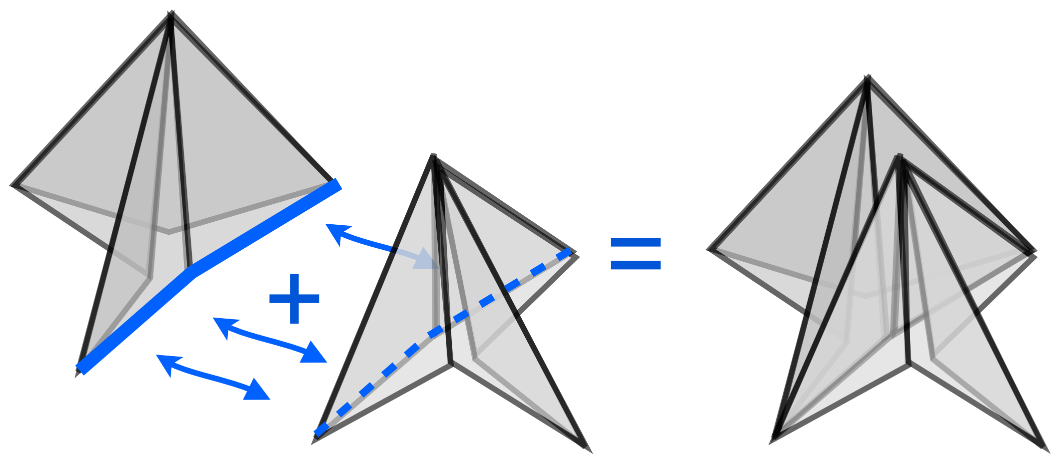


Gluing Operation 1

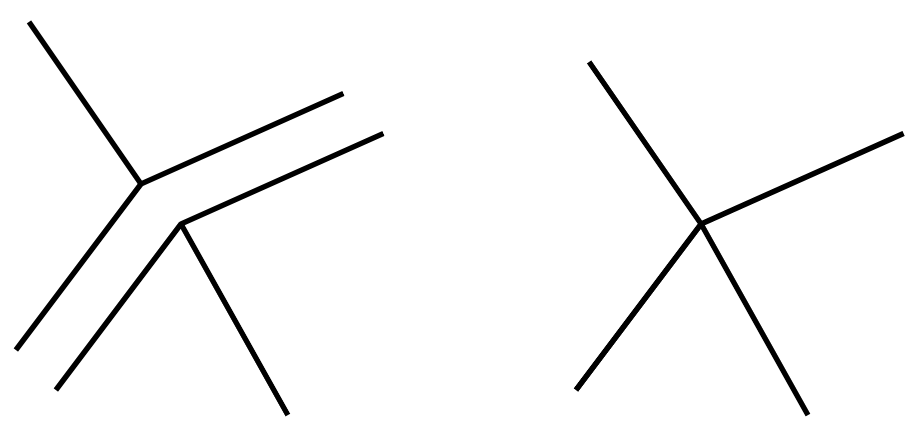


extension

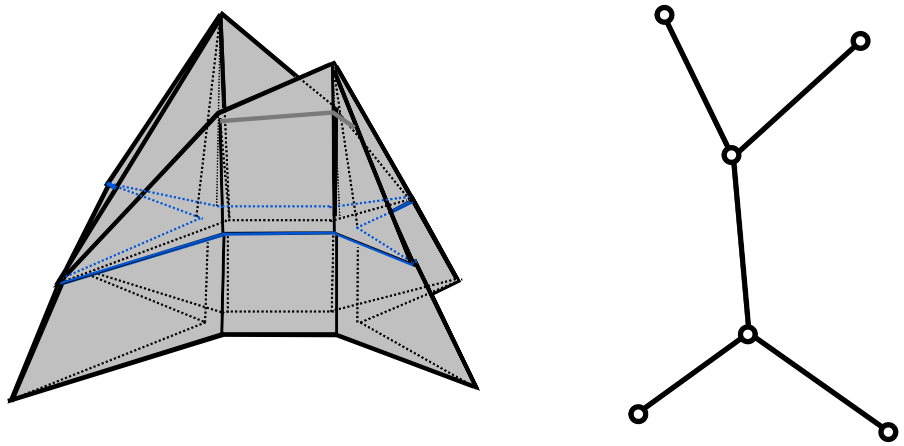
Gluing Operation 2

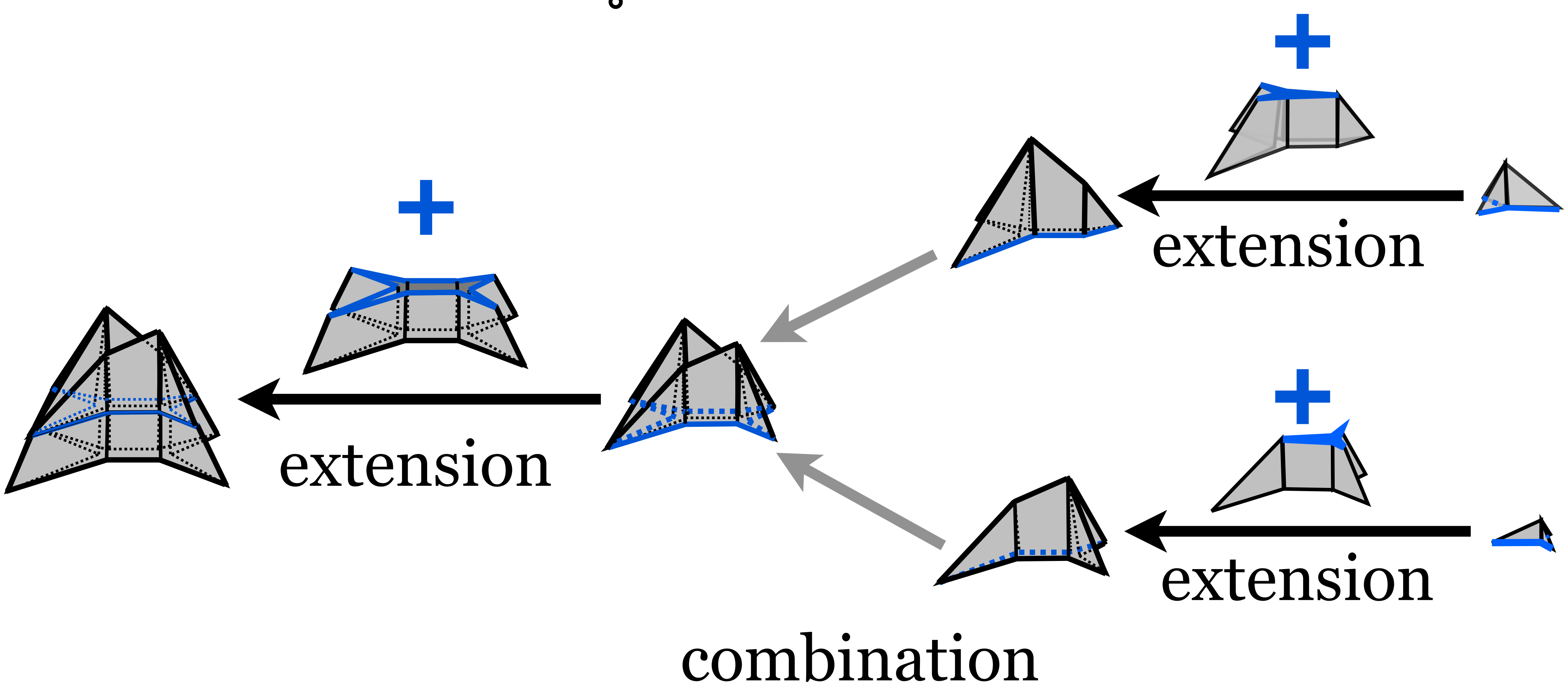


combination

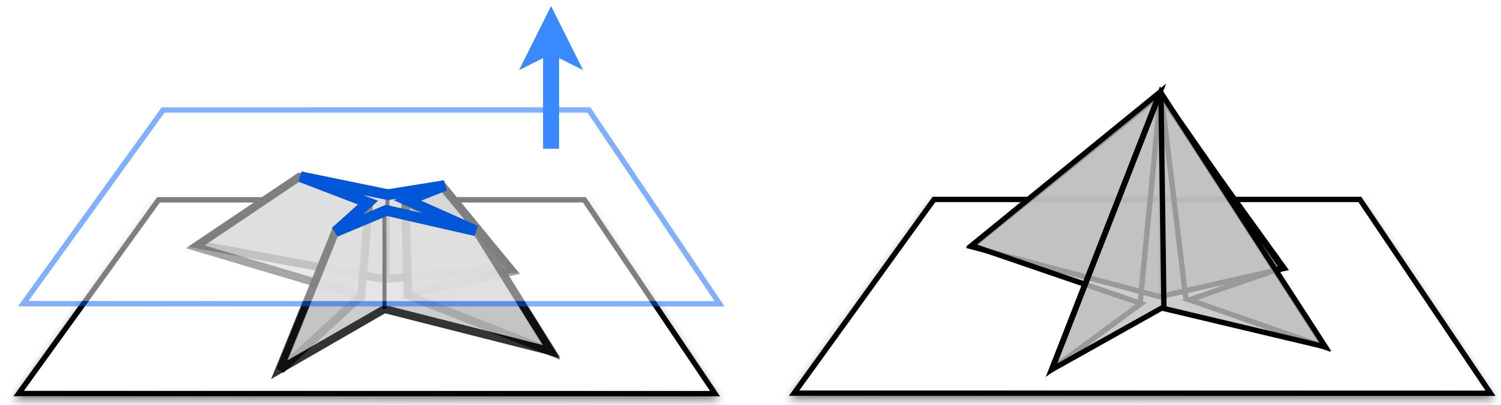


Lang Surface constructed on T

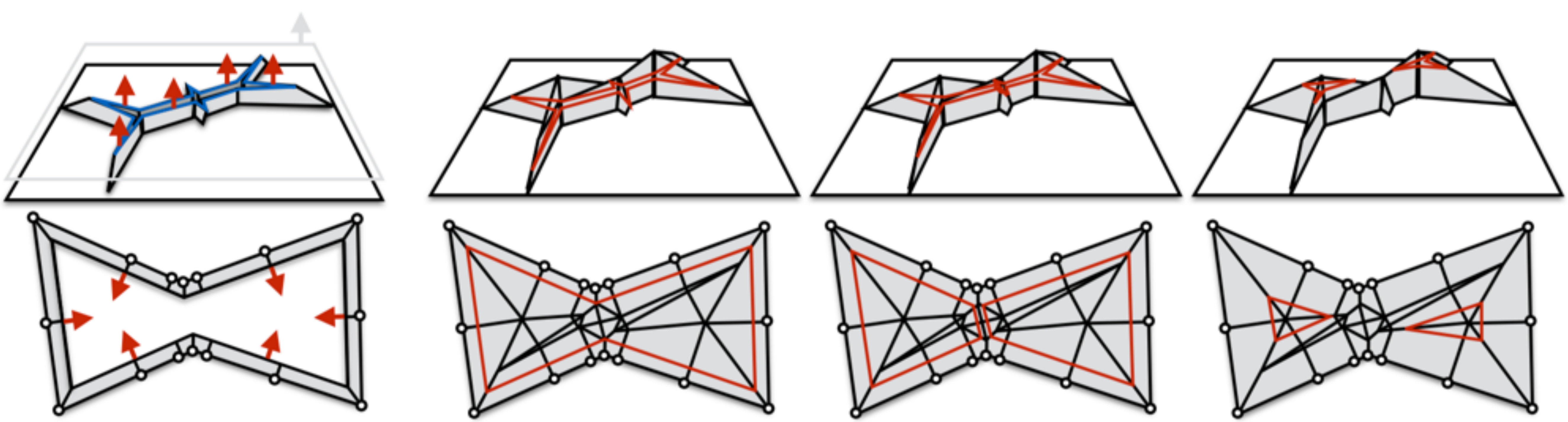




Recall:



- The **extrusion sweep:** replay of the sweep from the ground up.



(Informal) Result

The zero-curvature Lang surfaces with (intrinsically) convex boundary are the uniaxial bases produced by the Universal Molecule algorithm.

[Bowers and Streinu, 2014]

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- Generalization of Lang polygon and sweep to non-convex polygons (**Geodesic Lang polygons**).
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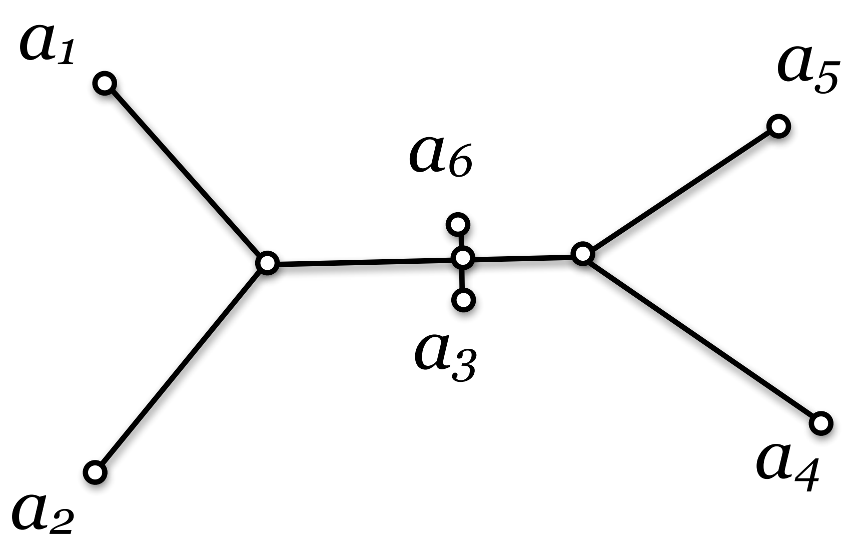
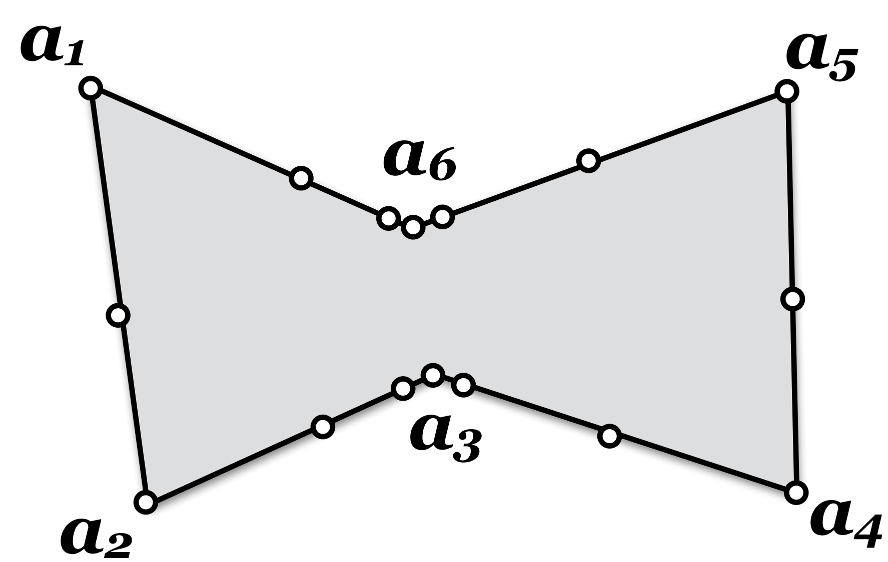
Geodesic Lang Polygons

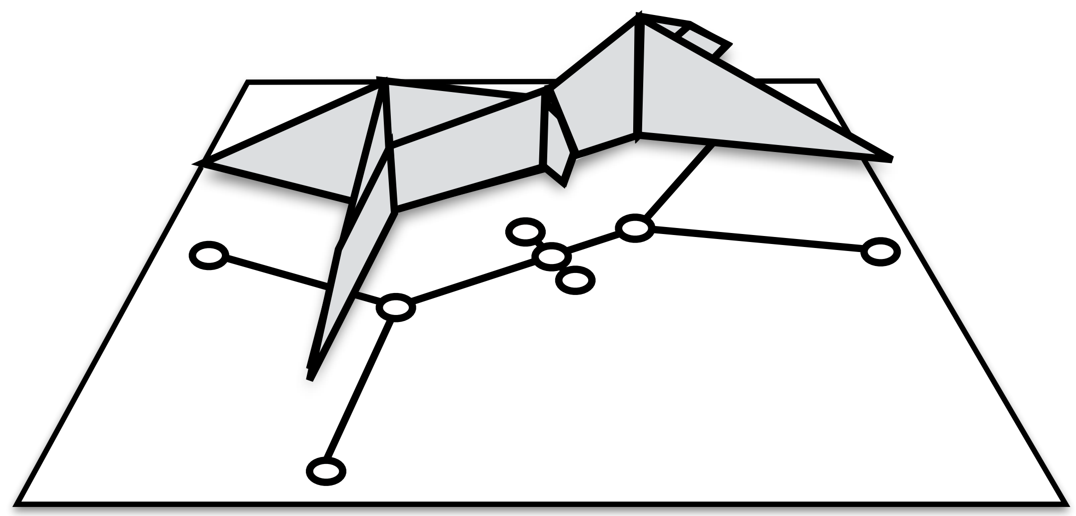


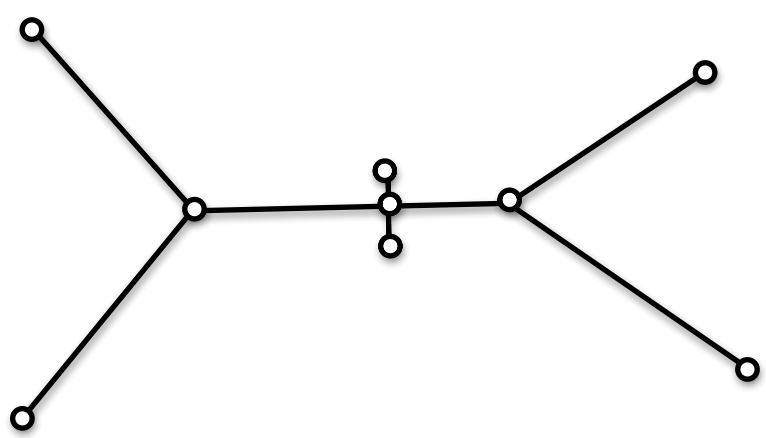
metric tree

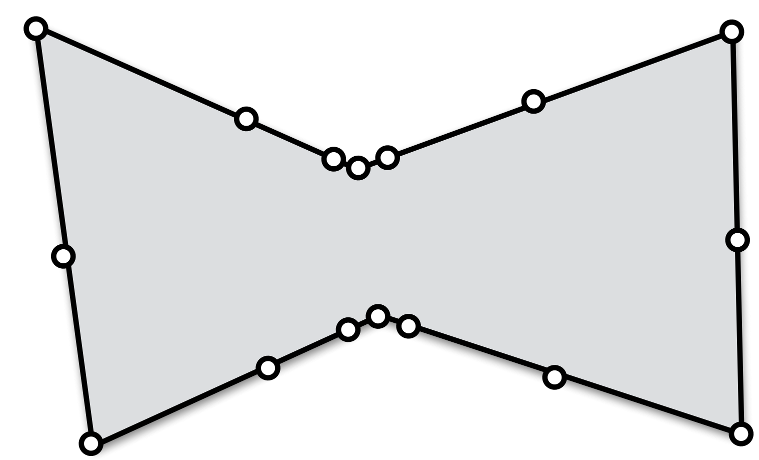


geodesic Lang polygon

1. Doubling polygon.
2. Geodesic distance between two vertices is greater than or equal to corresponding distance in tree. (**geodesic Lang property**)
3. Vertices corresponding to internal nodes are straight.

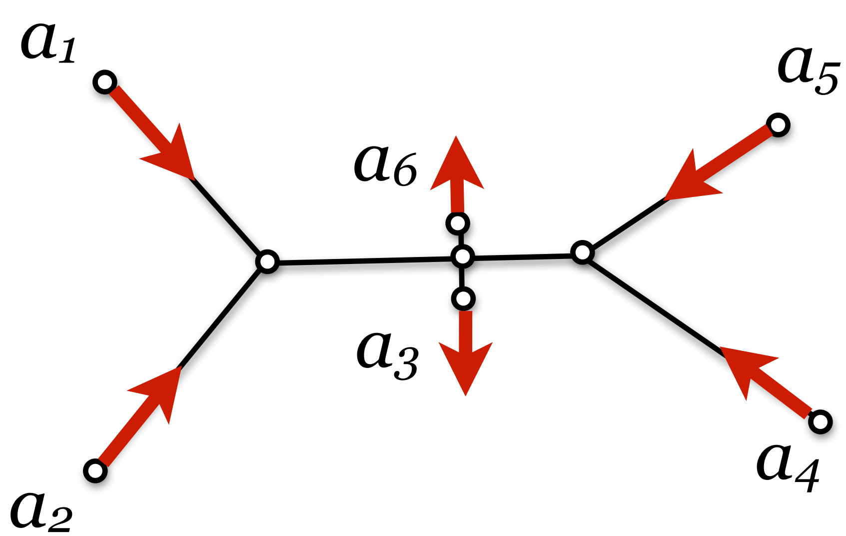


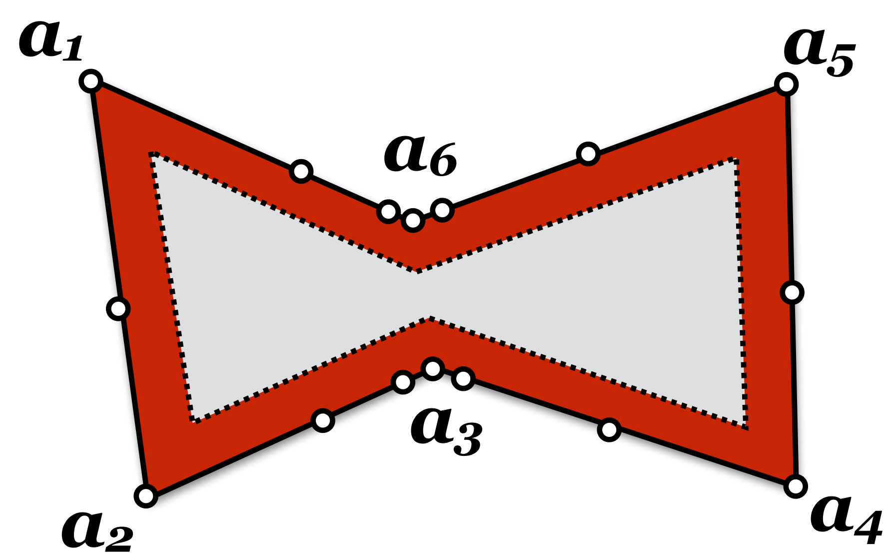




Boundary polygon of a flat Lang surface is a geodesic Lang polygon.

Generalize the Sweep





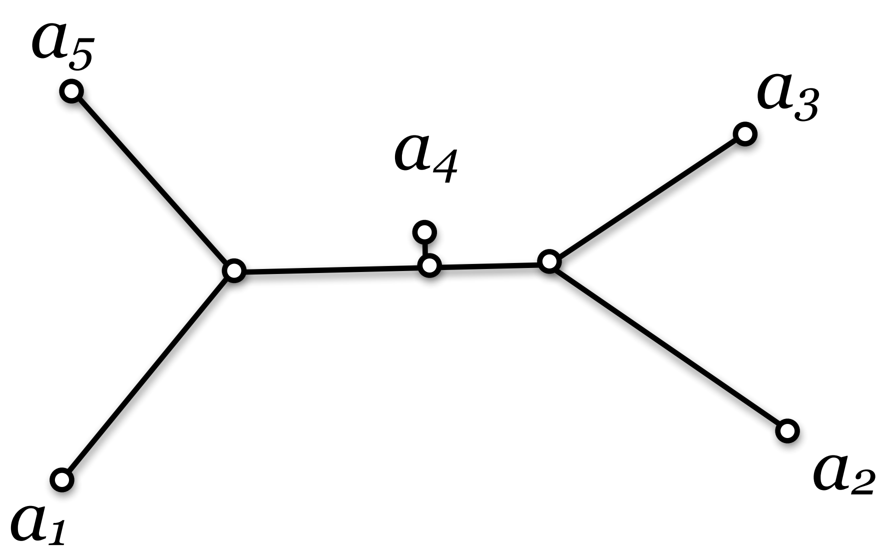
Sweep of a Lang polygon *(T, P_T):*

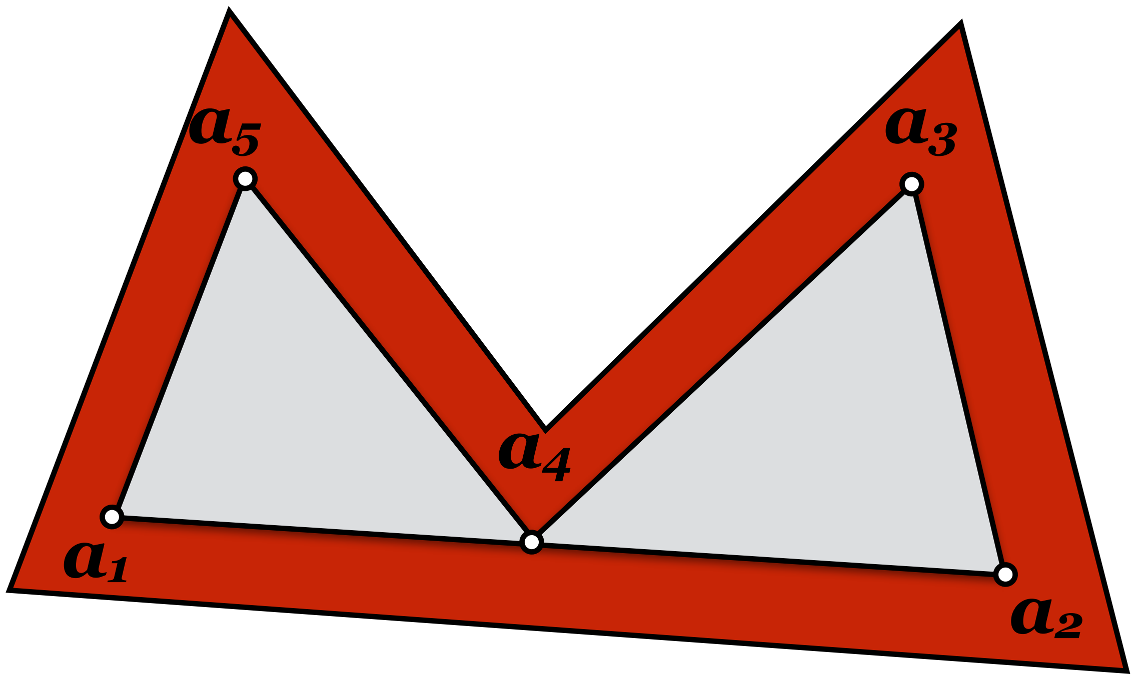
- Growing/shrinking process in the tree
- Must maintain the geodesic Lang property

May split into multiple polygon/tree pairs.

Important Properties

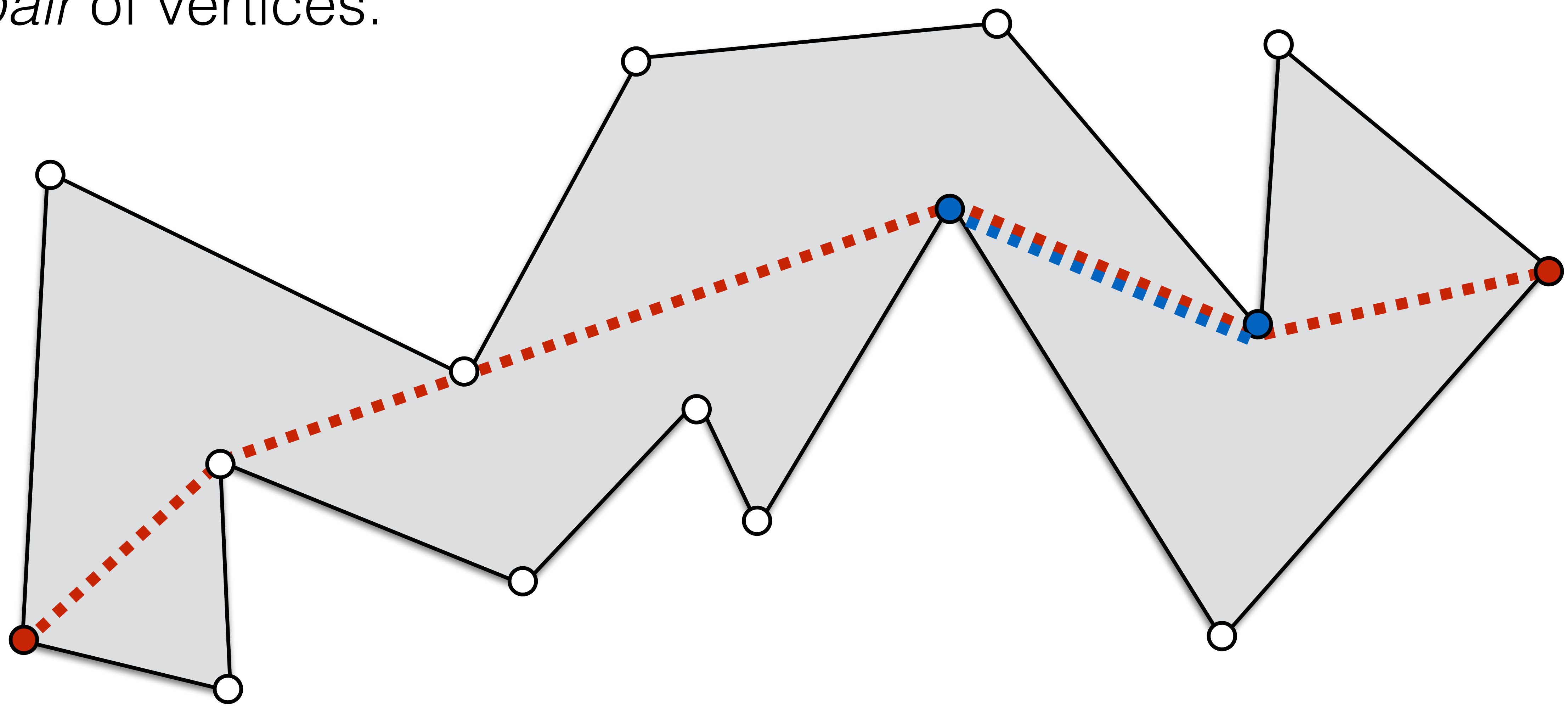
- Self-touch implies violation of geodesic Lang property



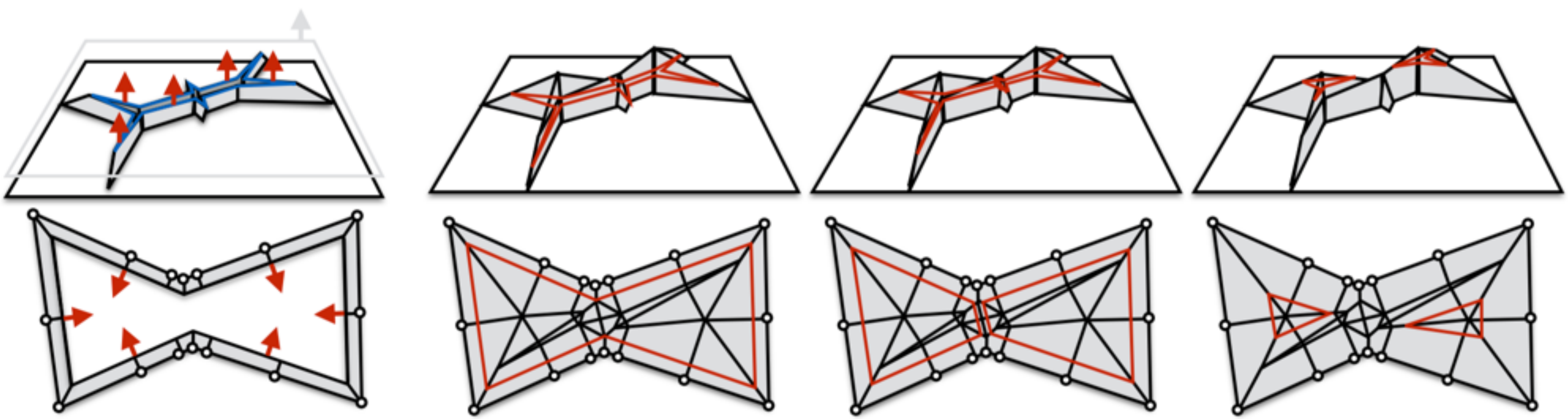


Important Properties

- If the geodesic distance is shorter than the tree distance for some pair of vertices, then this is also true for a *visible pair* of vertices.



- The **extrusion sweep** of a Lang surface is a generalized sweep of the polygon and tree that maintains the geodesic Lang property.



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Geodesic UM Algorithm

- **Input:** geodesic Lang polygon (T, P_T)
- **Output:** Crease pattern on P_T that is equivalent to a zero-curvature Lang surface.
- Simulate the sweep in both the polygon and tree.
- Split along visible pairs when the Lang property is violated.
- Naive Implementation:
  - Ignore the fact that a pair may or may not be visible, just compute for each pair when the euclidean version of the Lang property is violated.
  - Then test each potential splitting event for whether it constitutes a visible pair at the time of splitting.

Main Theorem

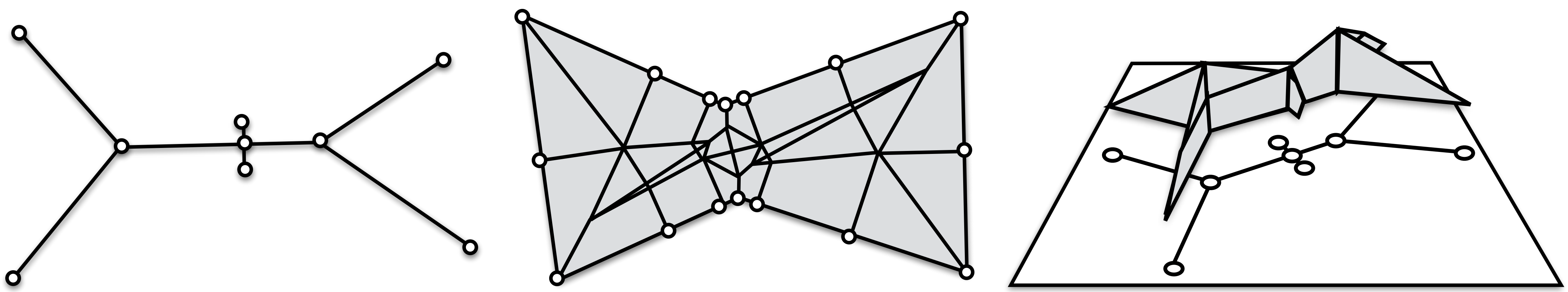
- Let P_T be a doubling-polygon for a tree T on a flat, disk-like piecewise-linear surface D. Then a Lang surface S constructed on T and isometric to P_T exists (and is unique) if and only if P_T is a geodesic doubling-polygon for T on D.

[Bowers and Streinu, Submitted]

Thank You
& Questions?

Proof Highlights

- Given a geodesic Lang polygon *(T, P_T)*
  - (1) there exists a Lang surface S constructed on T whose boundary polygon is P_T and
  - (2) the subdivision of P_T given by the geodesic universal molecule crease pattern is the same as the subdivision of S into vertices, edges, and faces.



- There is only one generalized sweep of a Lang polygon that maintains the geodesic Lang property.

Boundary Curvature