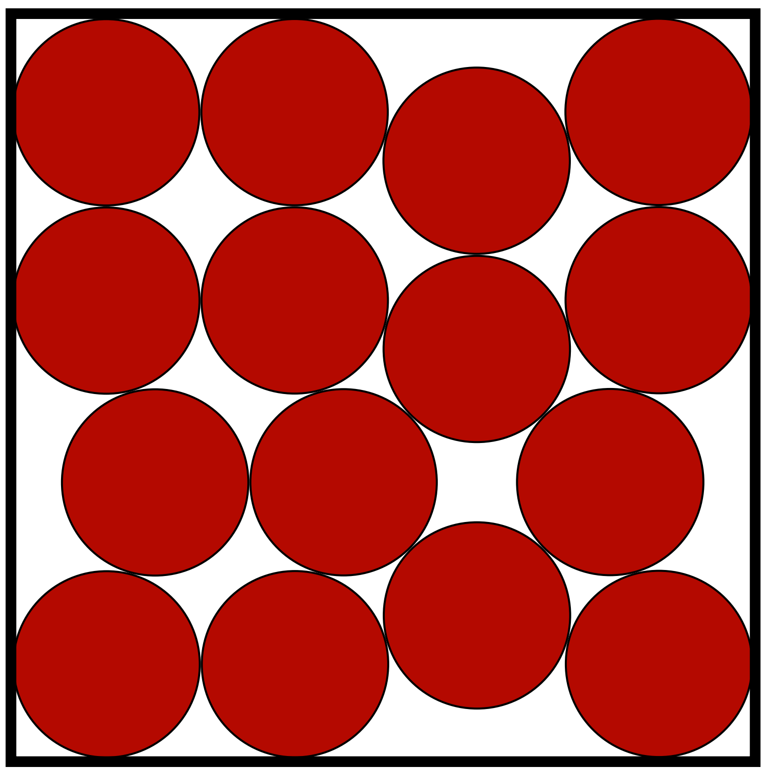
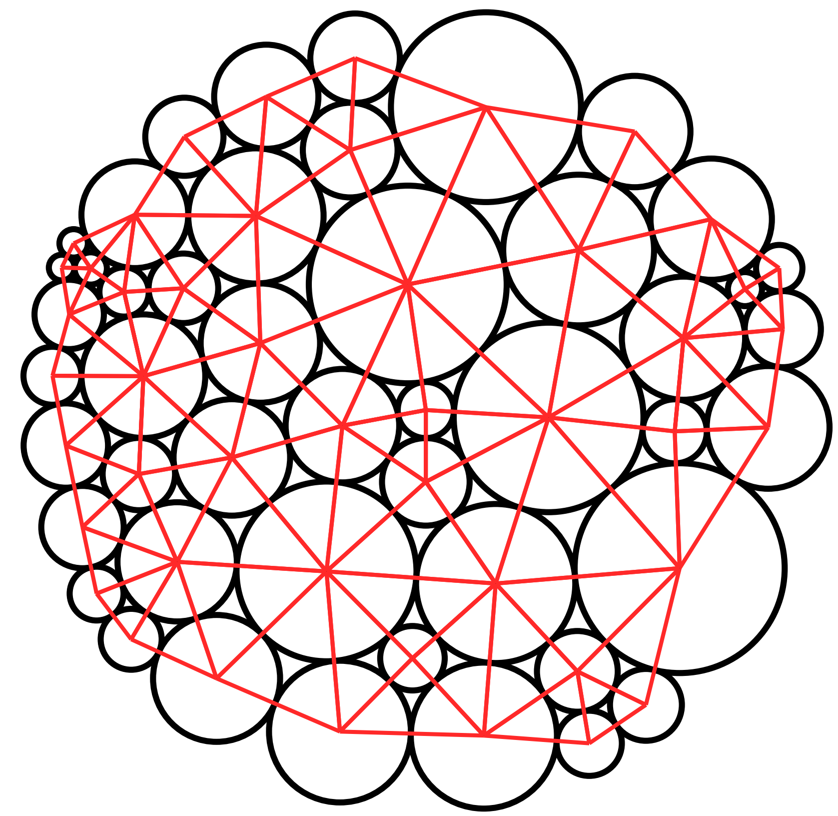
Non-globally Rigid Inversive Distance Circle Packings

John C. Bowers
Dept. of Computer Science
James Madison University

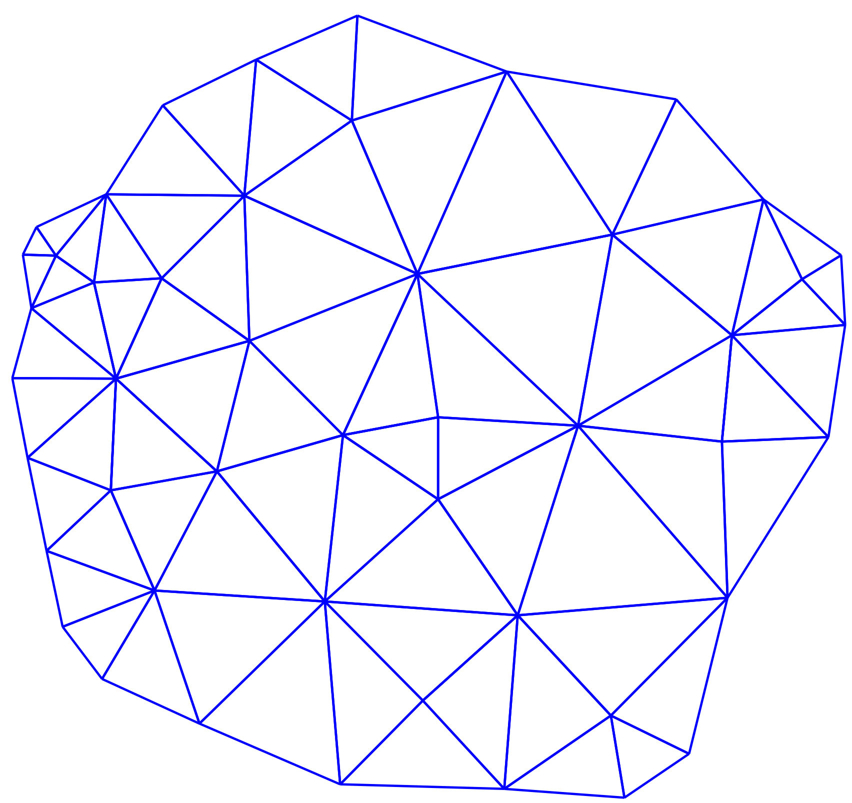
Joint work with:
Philip L. Bowers
Dept. of Mathematics
The Florida State University

Which Circle Packing?

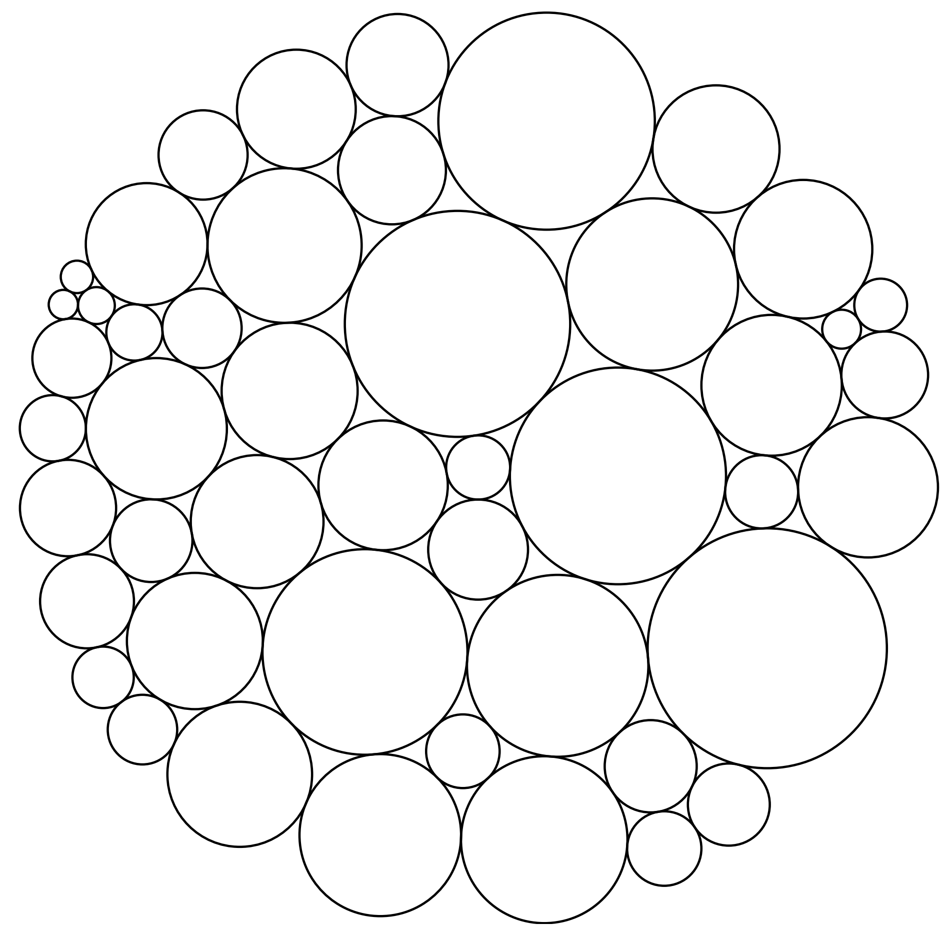
not this:	this:
	
• Not necessarily a triangulation • Combinatorics can be variable. • Radii are fixed. • Has 3D “sphere packing” analog	• Triangulation • Combinatorics are fixed. • Radii are variable. • No “sphere packing” analog

Circle Packing Defn.

- Given a triangulation *T*, a ***circle packing*** is a configuration of circles *P* whose tangency pattern is *T*.



T

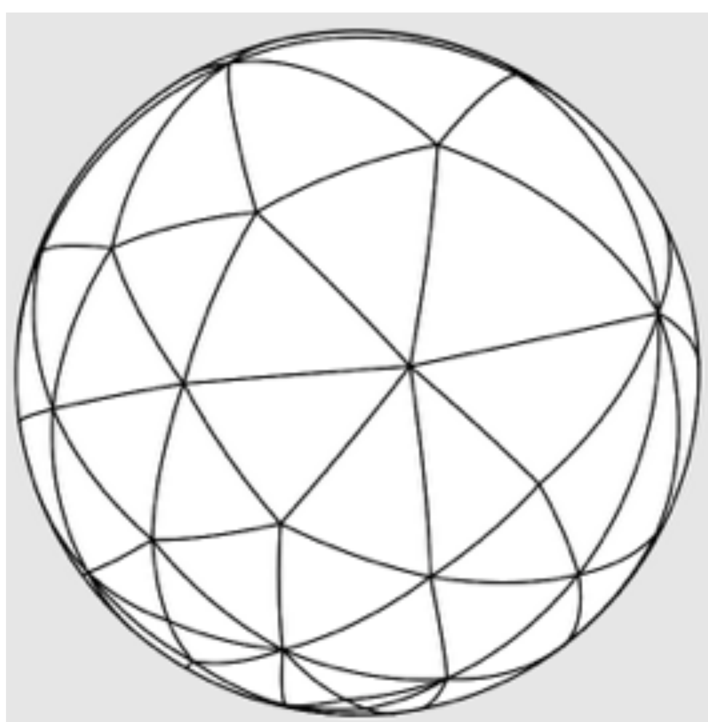


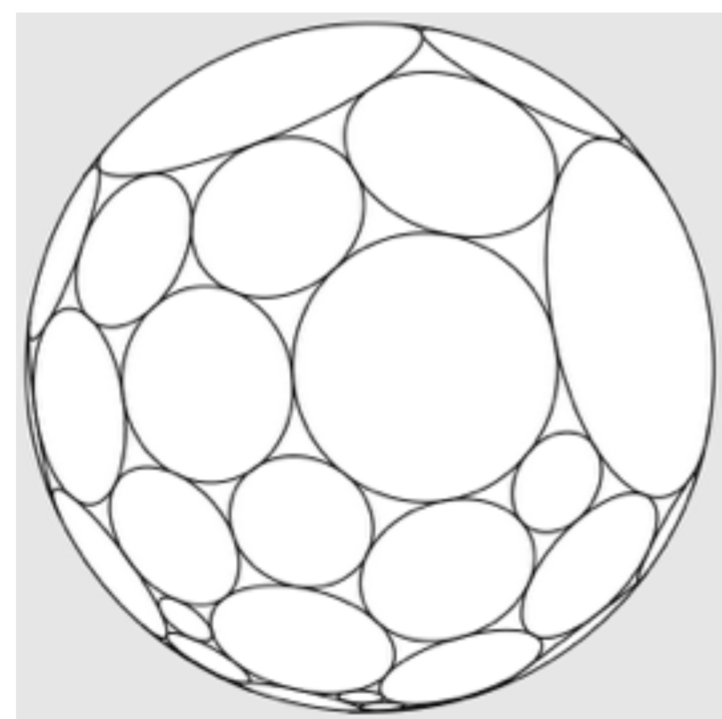
P

Koebe-Adreev-Thurston Theorem

- **Theorem** (Koebe-Adreev-Thurston): Given any triangulation *T* of a topological sphere, there exists a *univalent* circle packing *P* of the Riemann sphere having the same combinatorics as *T*. **Furthermore**, *P* is **unique** up to Möbius transformations and inversions.

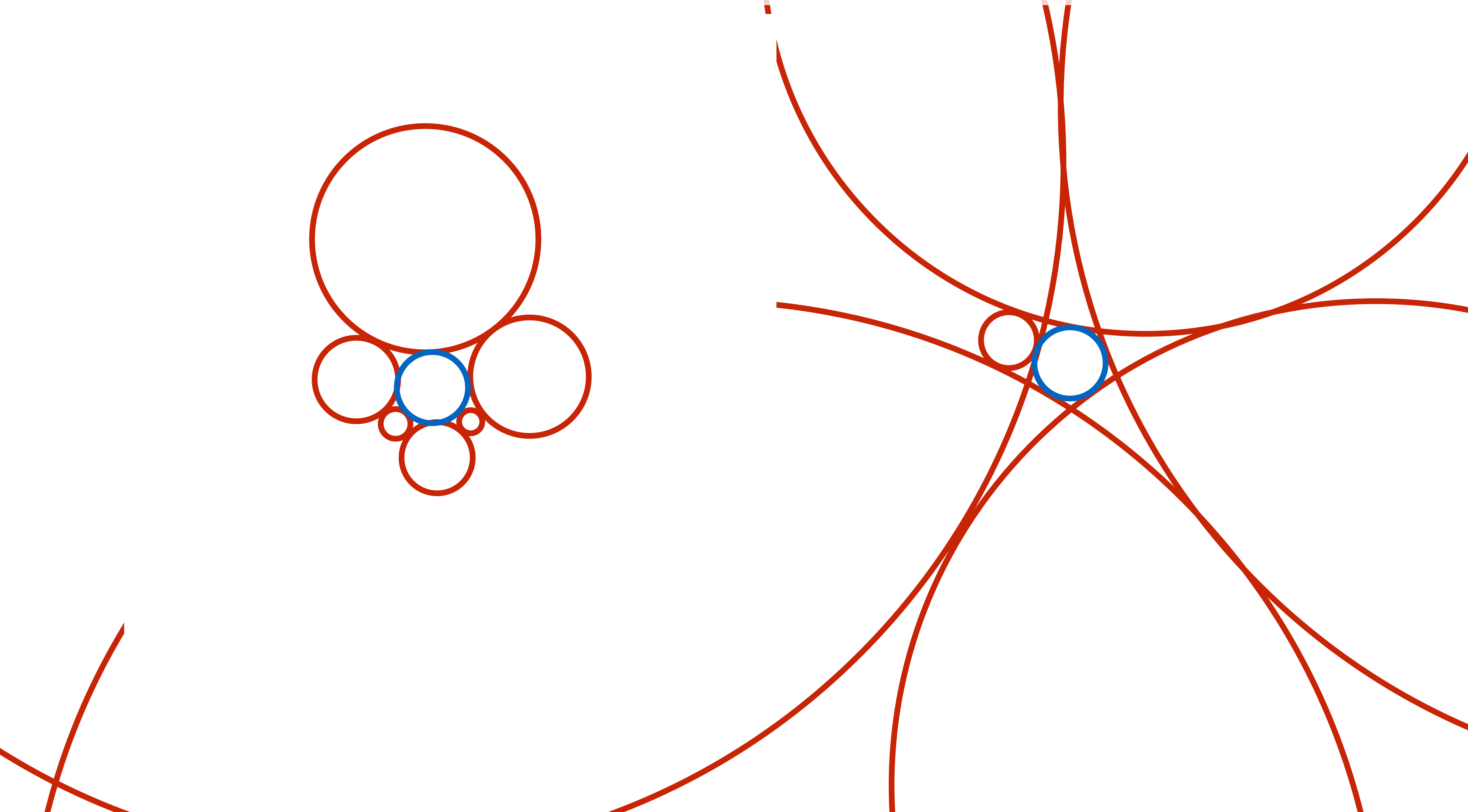
  Alternatively: Circle packings of the sphere *exist* and are **globally rigid**.



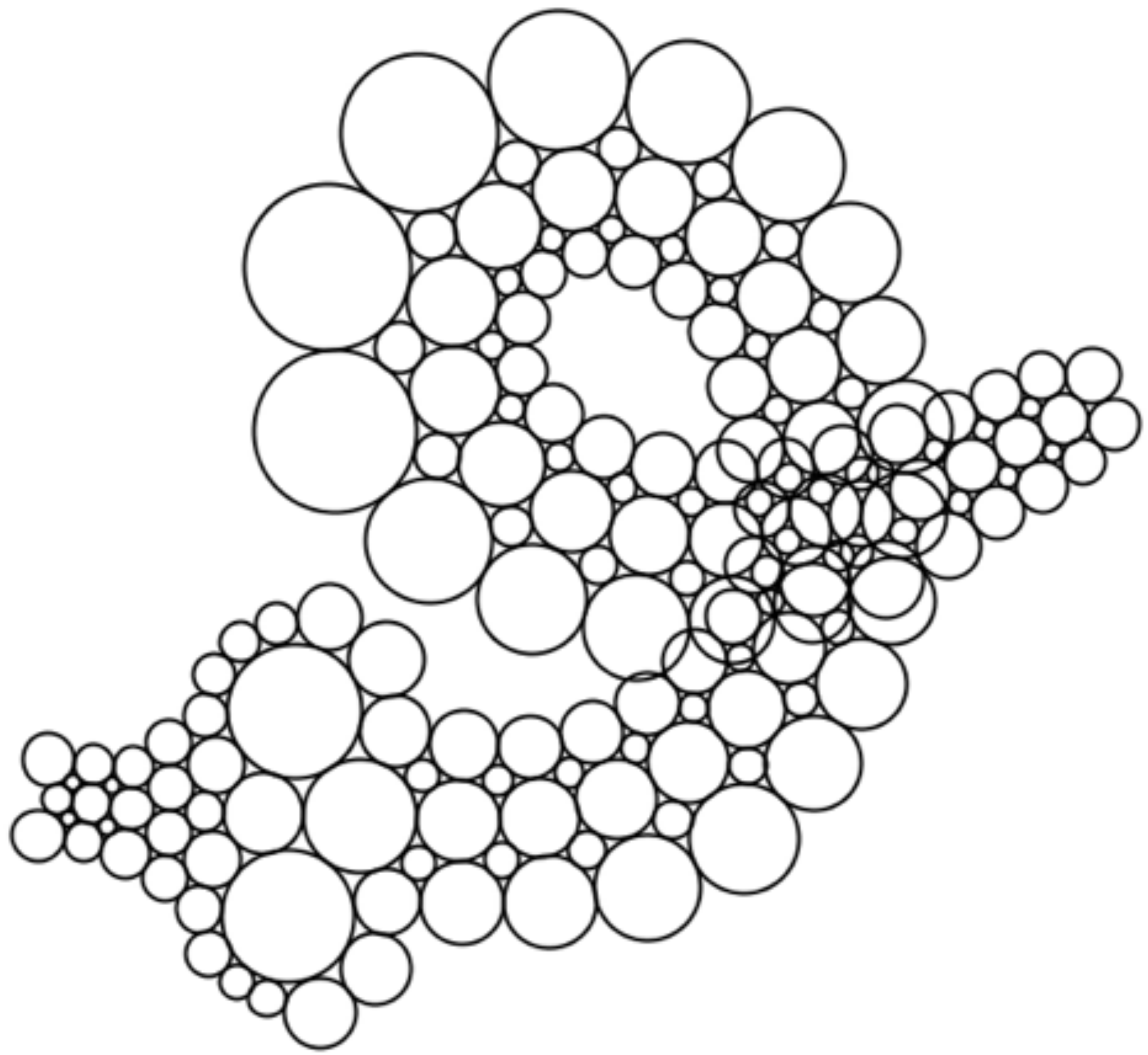


Extended to closed surfaces by Thurston and the disk by Beardon and Stephenson

Univalence



Univalence is Local



snake.p packing from http://www.math.utk.edu/~kens/CirclePack/

Inversive Distance

- Formula in $\mathbb{E}^{2}$:

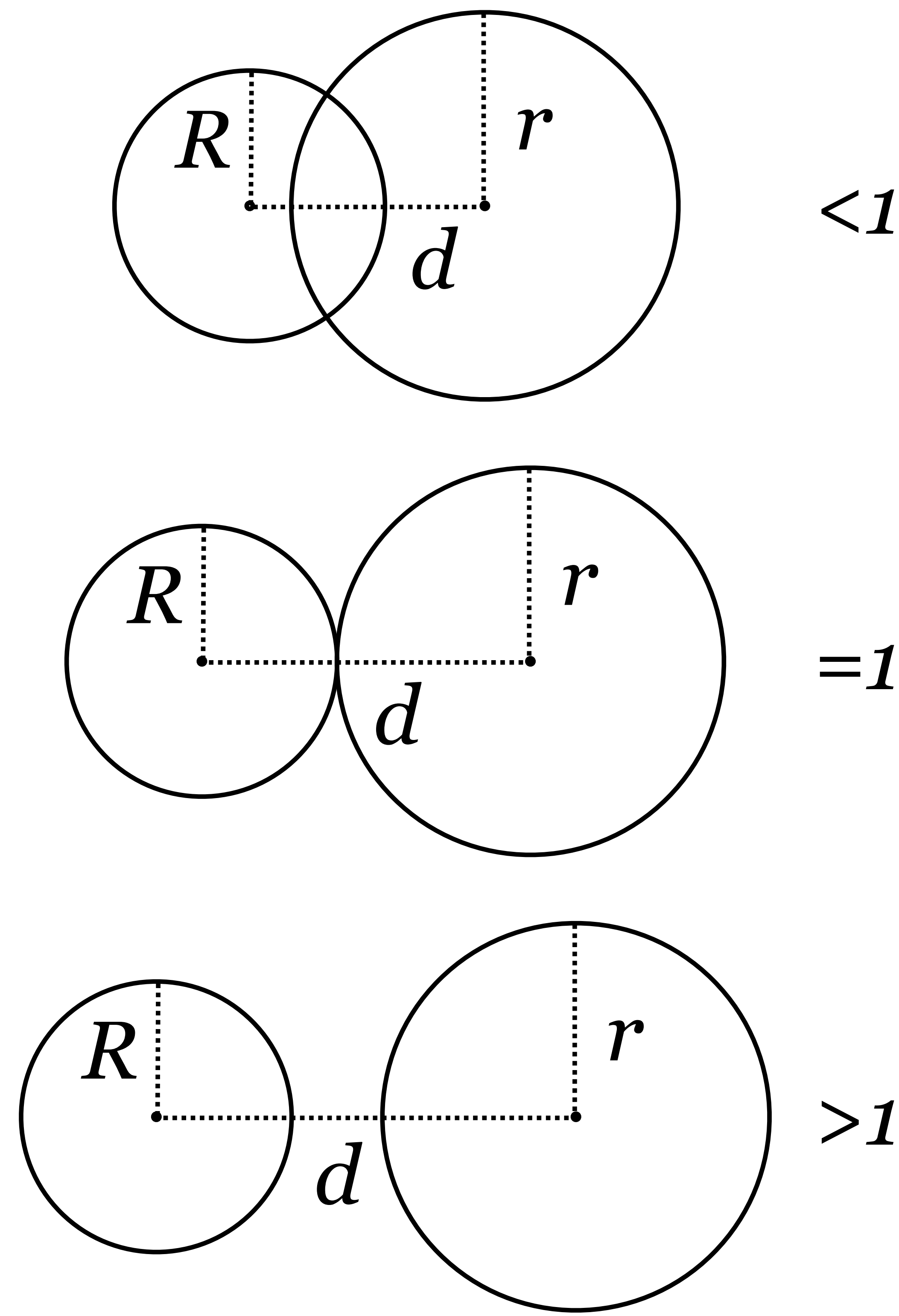
$$Inv(C_1,C_2) = \frac{d^2-R^2-r^2}{2Rr}$$

  Also defined in sphere and hyperbolic spaces.

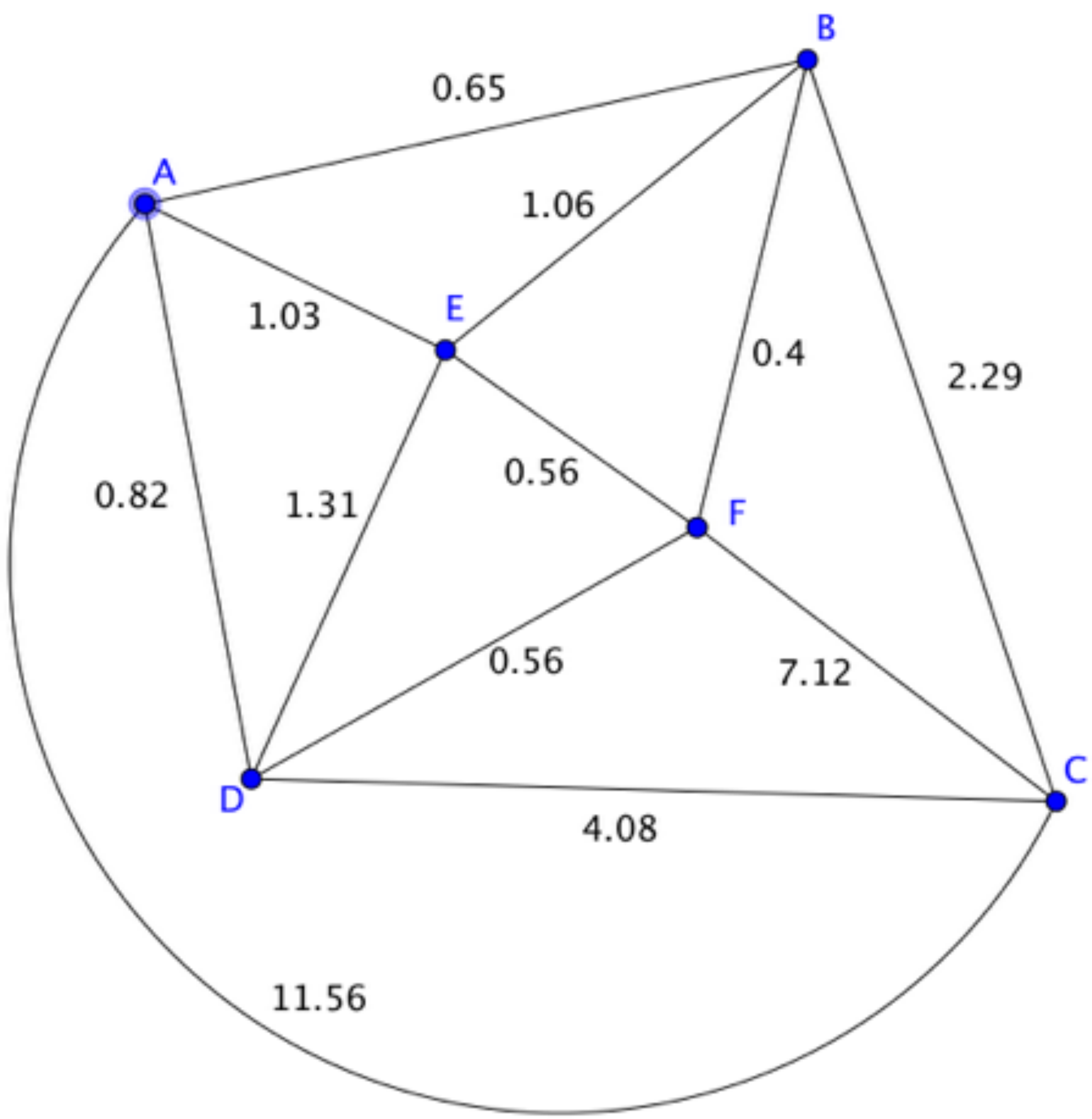
- Acts as a "Distance" between circles.

- **Not** a metric:
  - *Inv(C, C)* is not 0
  - Does *not* satisfy Δ-inequality
  - Takes negative values

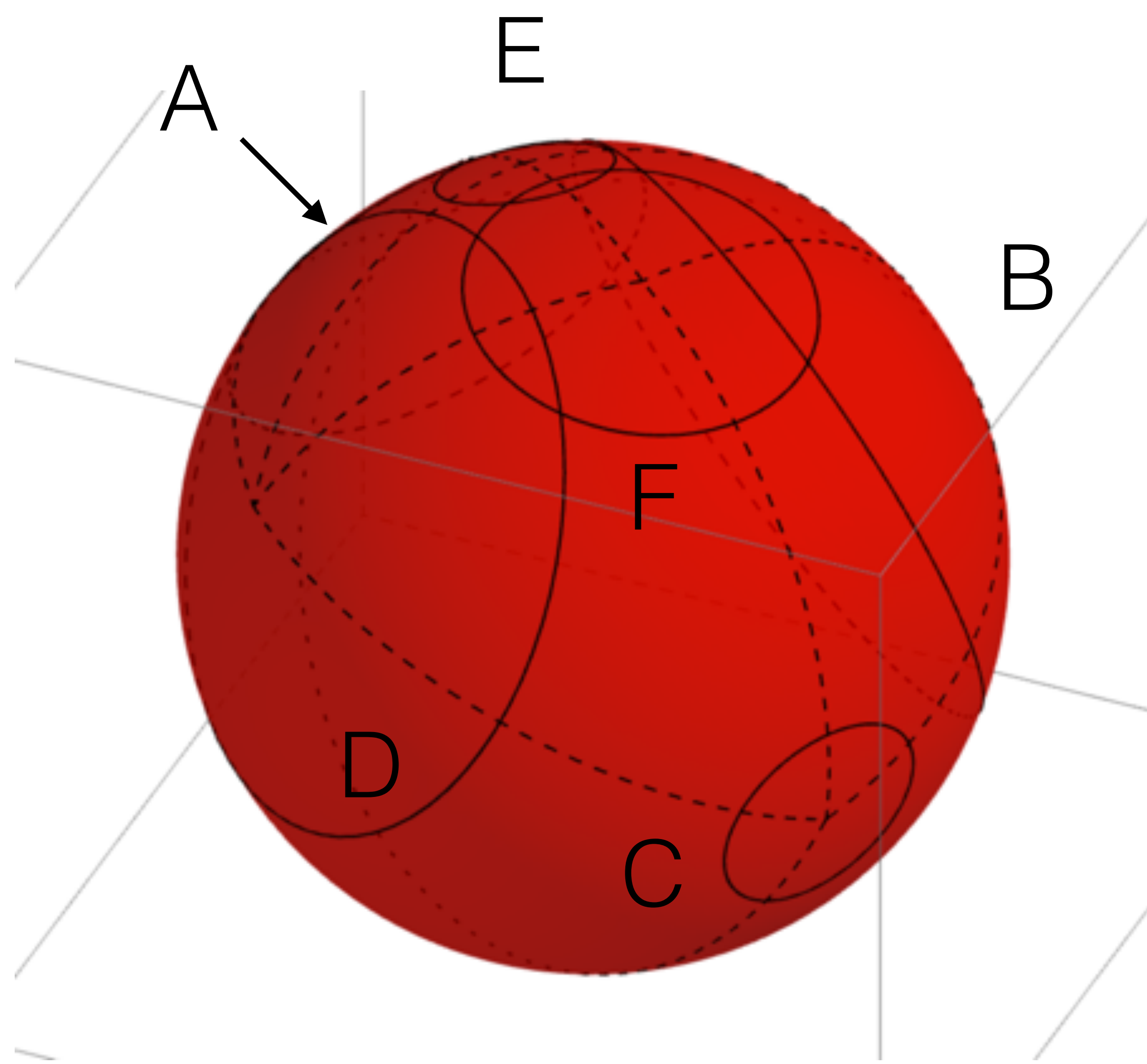
- Invariant under:
  - Möbius transformations (on the sphere, mostly true in $\mathbb{E}^2$)
  - Stereographic projections



Inversive Distance Circle Packings

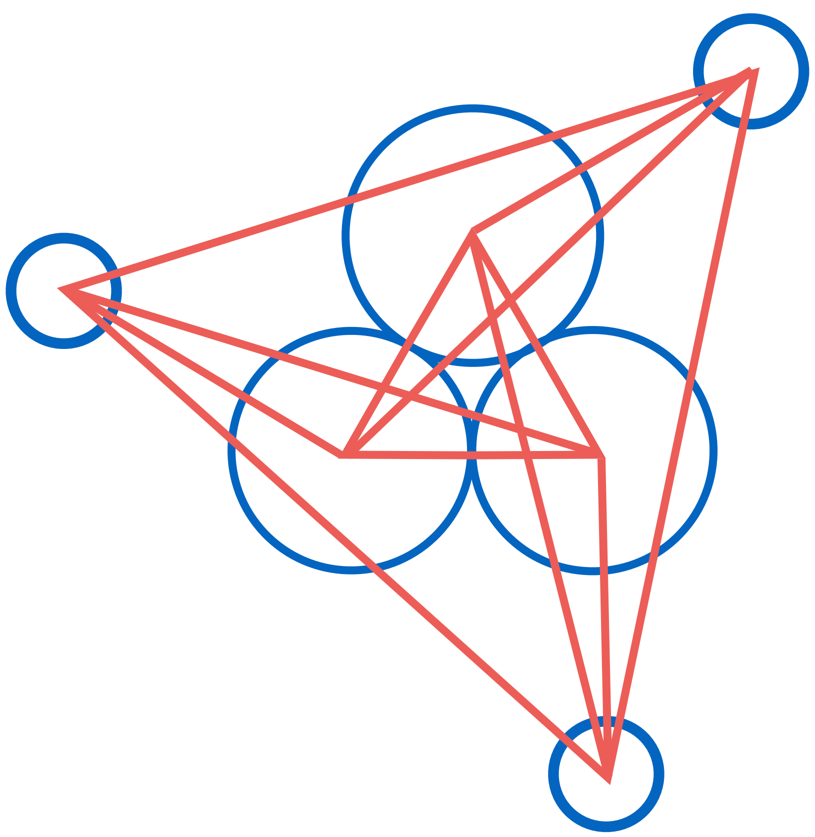


labeled octahedral graph

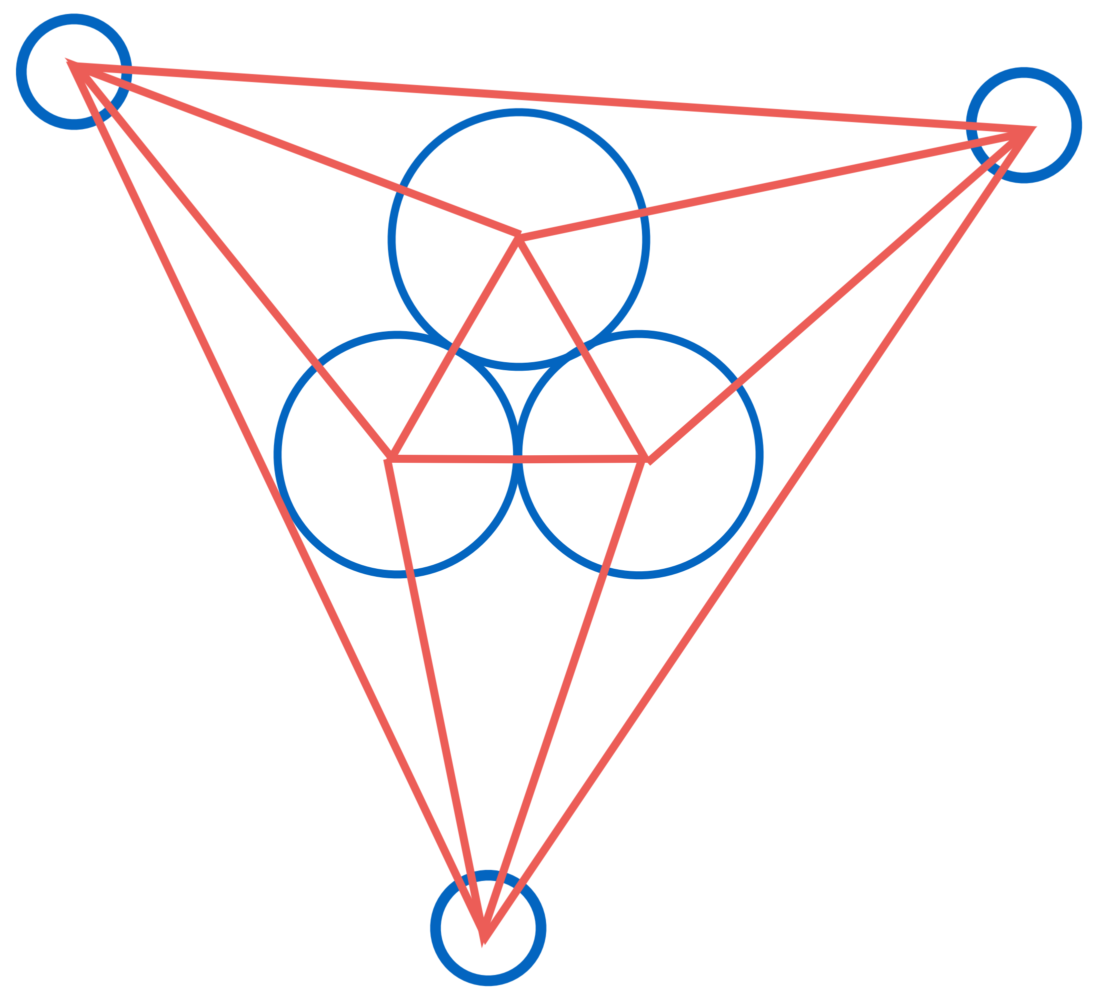


inversive distance
circle packing

Realization vs. Packing



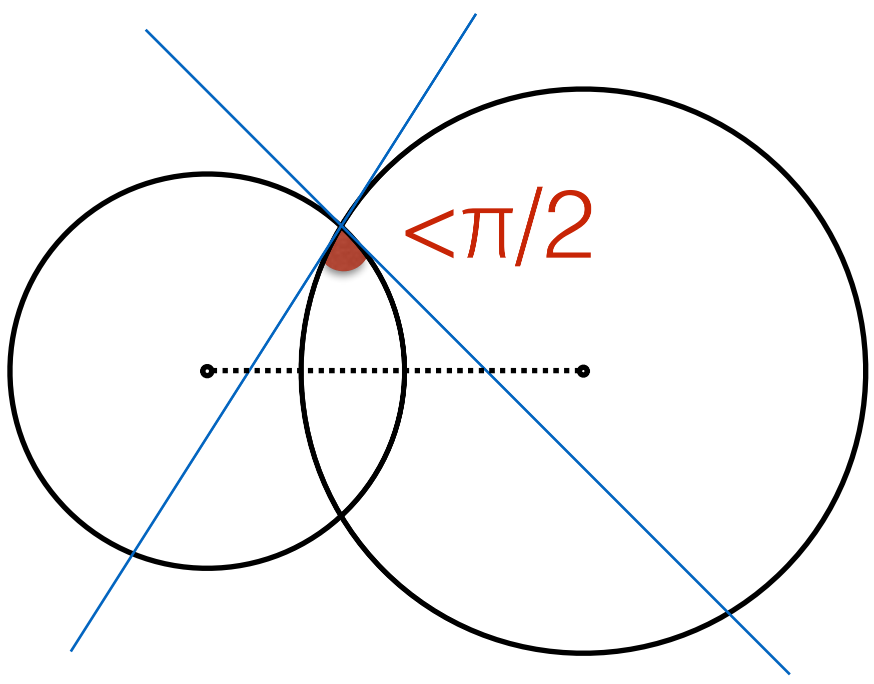
circle realization



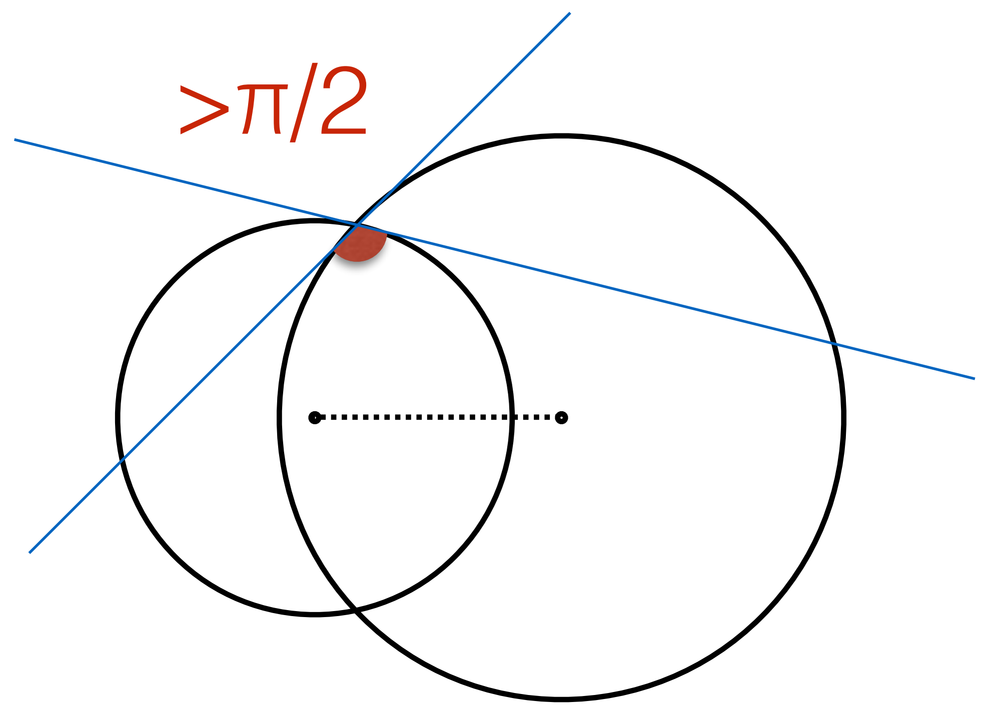
circle packing

(Edge-)Segregated Packings

Inversive Distance > 0



Inversive Distance < 0



Bowers-Stephenson Question

- Given a triangulation of a closed surface, concerns the uniqueness of *segregated inversive distance* circle packings.
  - On the torus: are they unique up to Euclidean scaling + rigid transformations?

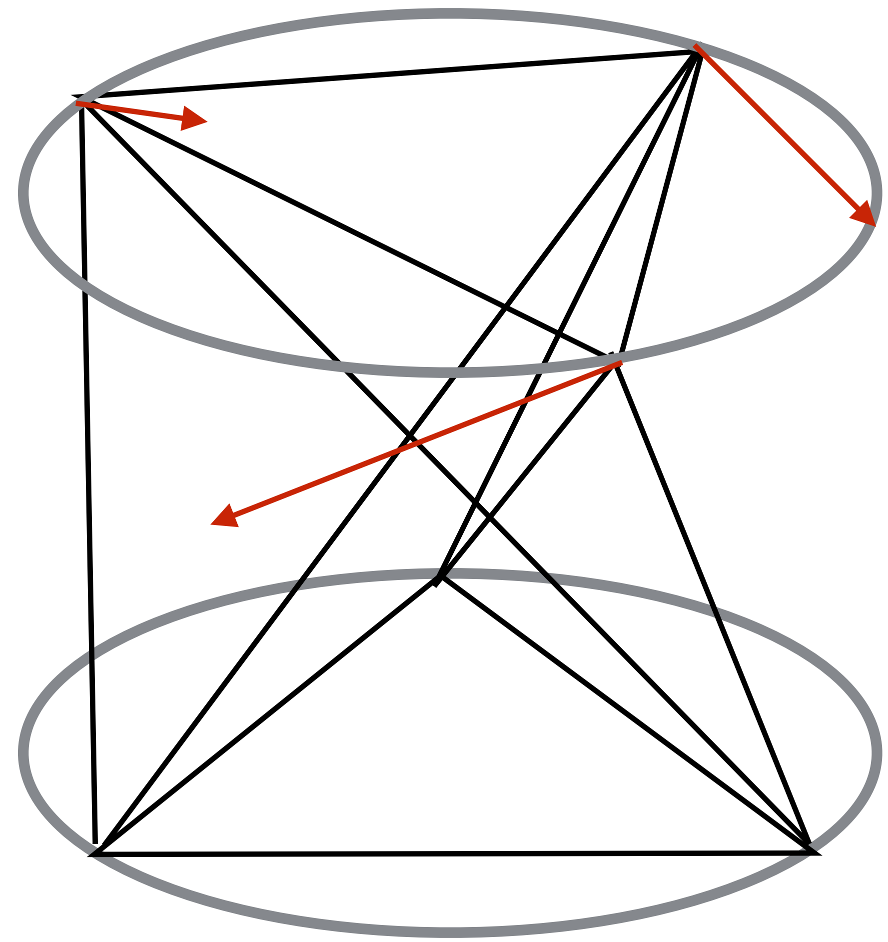
    Yes. Local Rigidity [Guo] Global Rigidity [Luo]
  - On closed hyperbolic surfaces: are they unique up to hyperbolic isometries?

    Yes. Local Rigidity [Guo] Global Rigidity [Luo]
  - On the 2-sphere: are they unique up to Möbius transformations and inversions?

    Not globally rigid! [Ma & Schlenker]

Ma-Schlenker Example

- Start with an infinitesimally flexible hyperideal Euclidean polyhedron.
- Use the infinitesimal flex to generate two hyperideal polyhedra that have the same edge lengths but are not equivalent.
- Convert the polyhedra to hyperbolic polyhedra.
- Use de Sitter space and Pogorelov maps to produce two non-Möbius equivalent inversive distance circle packings on the sphere.

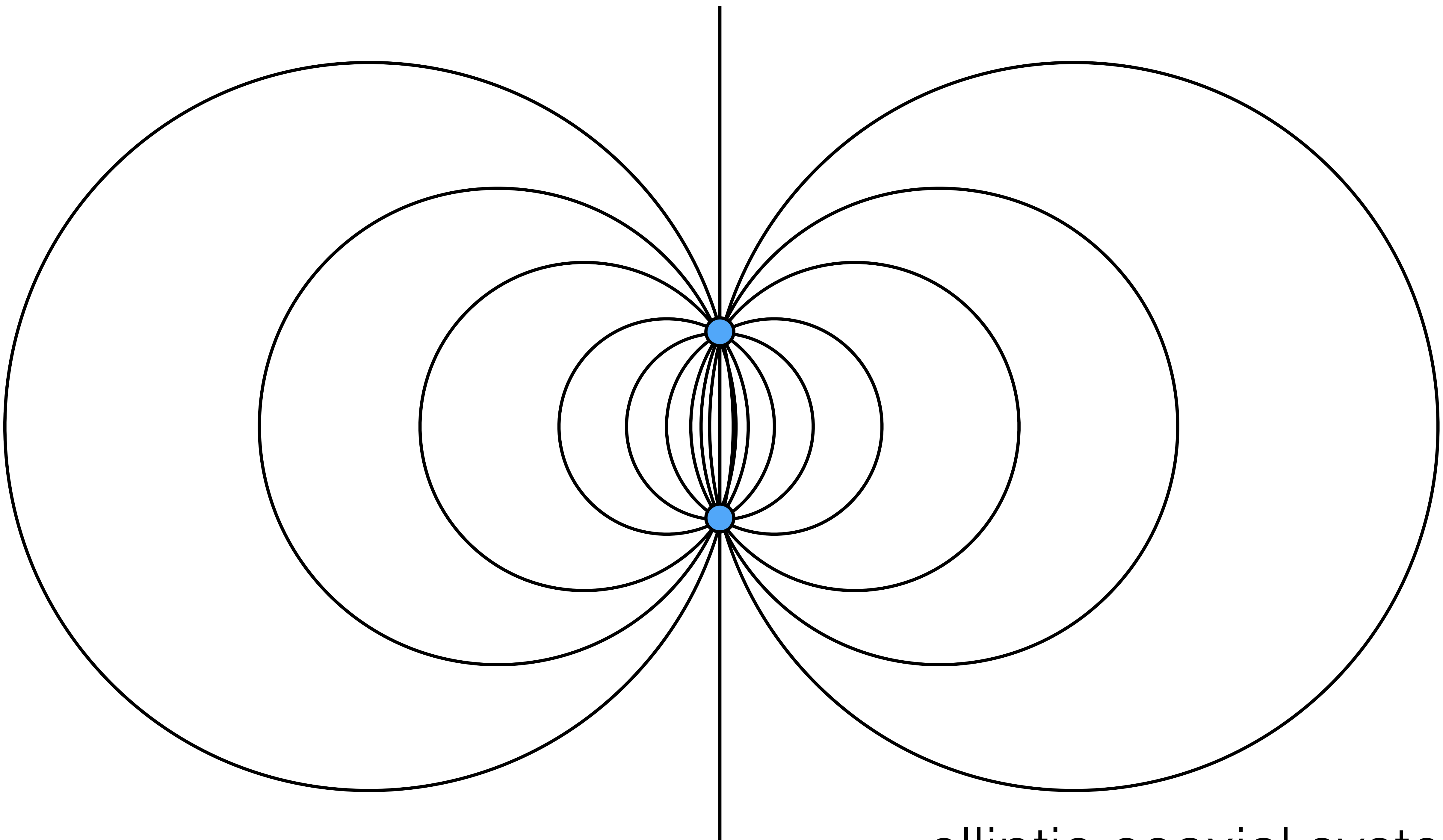


Schönhardt's twisted octahedron

Our goal: to construct examples like this intrinsically on the sphere using only inversive geometry.

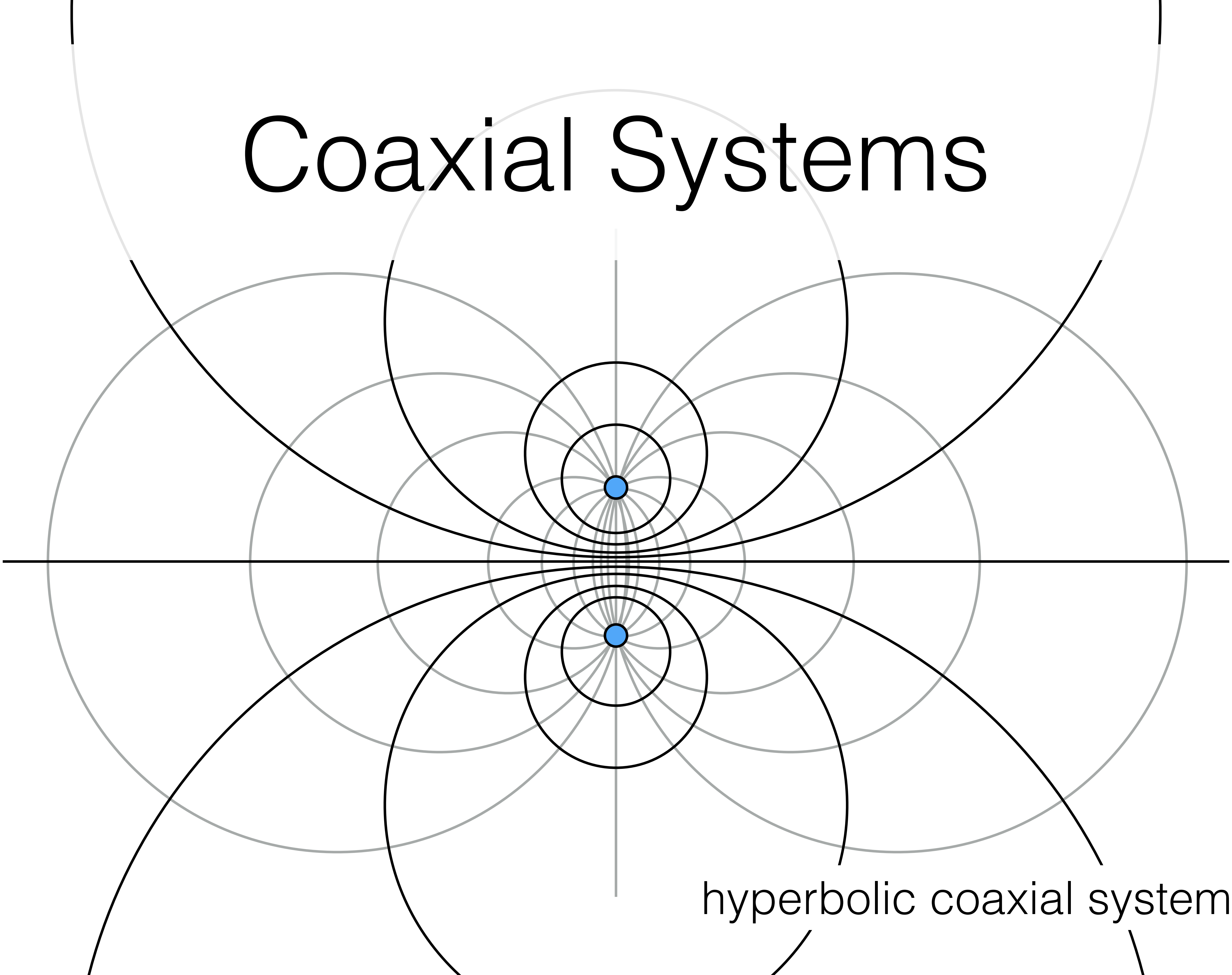
Inversive Geometry

Coaxial Systems



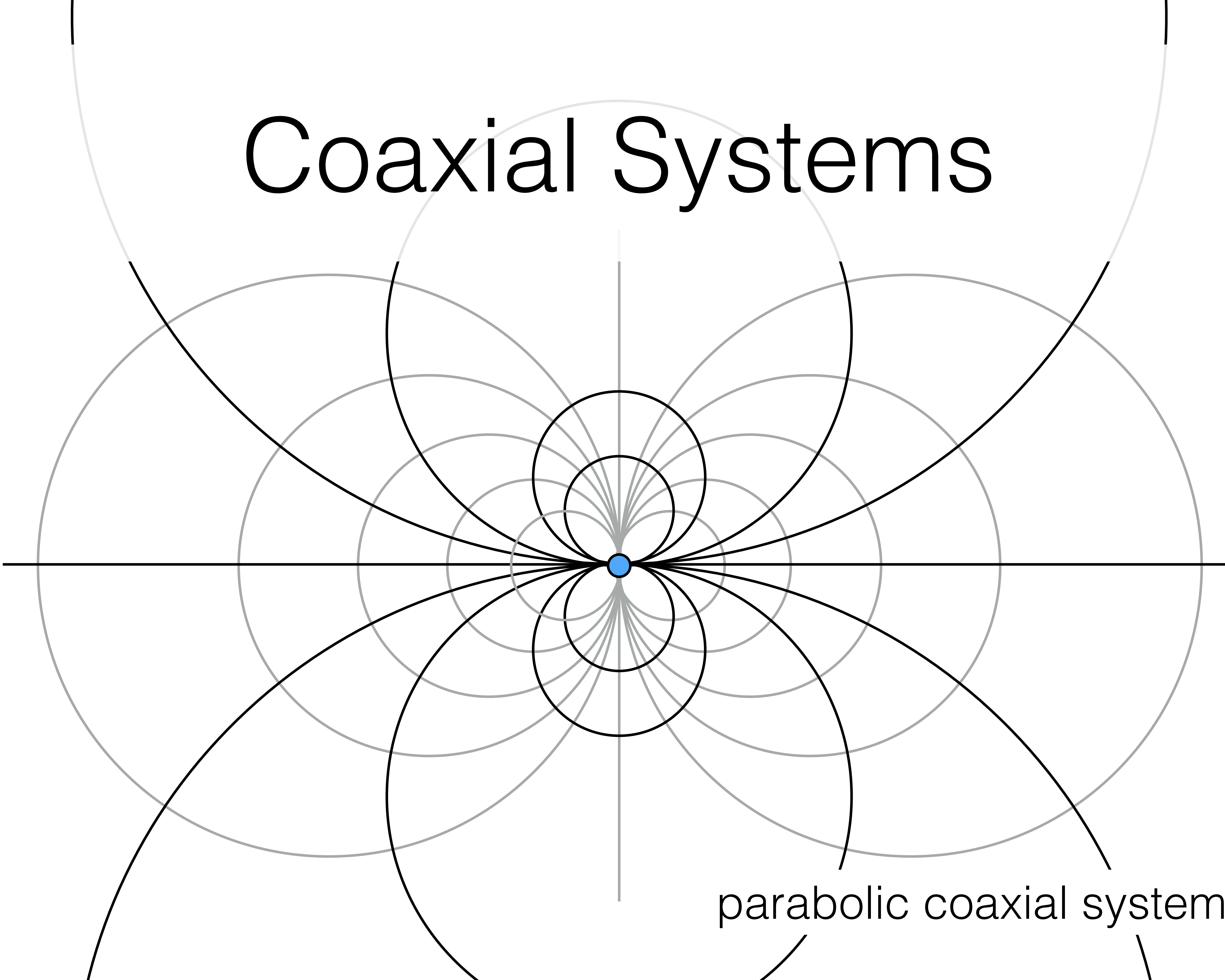
elliptic coaxial system

Coaxial Systems



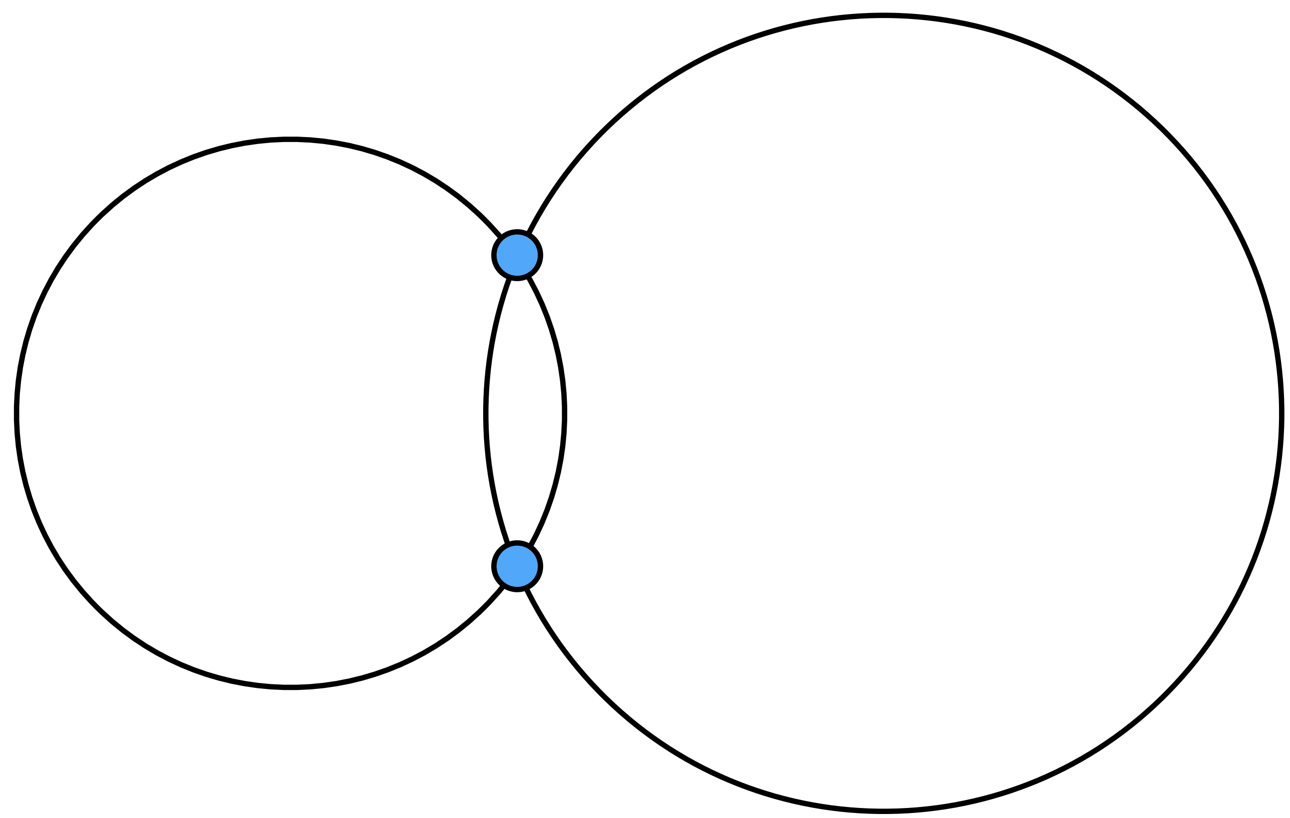
hyperbolic coaxial system

Coaxial Systems



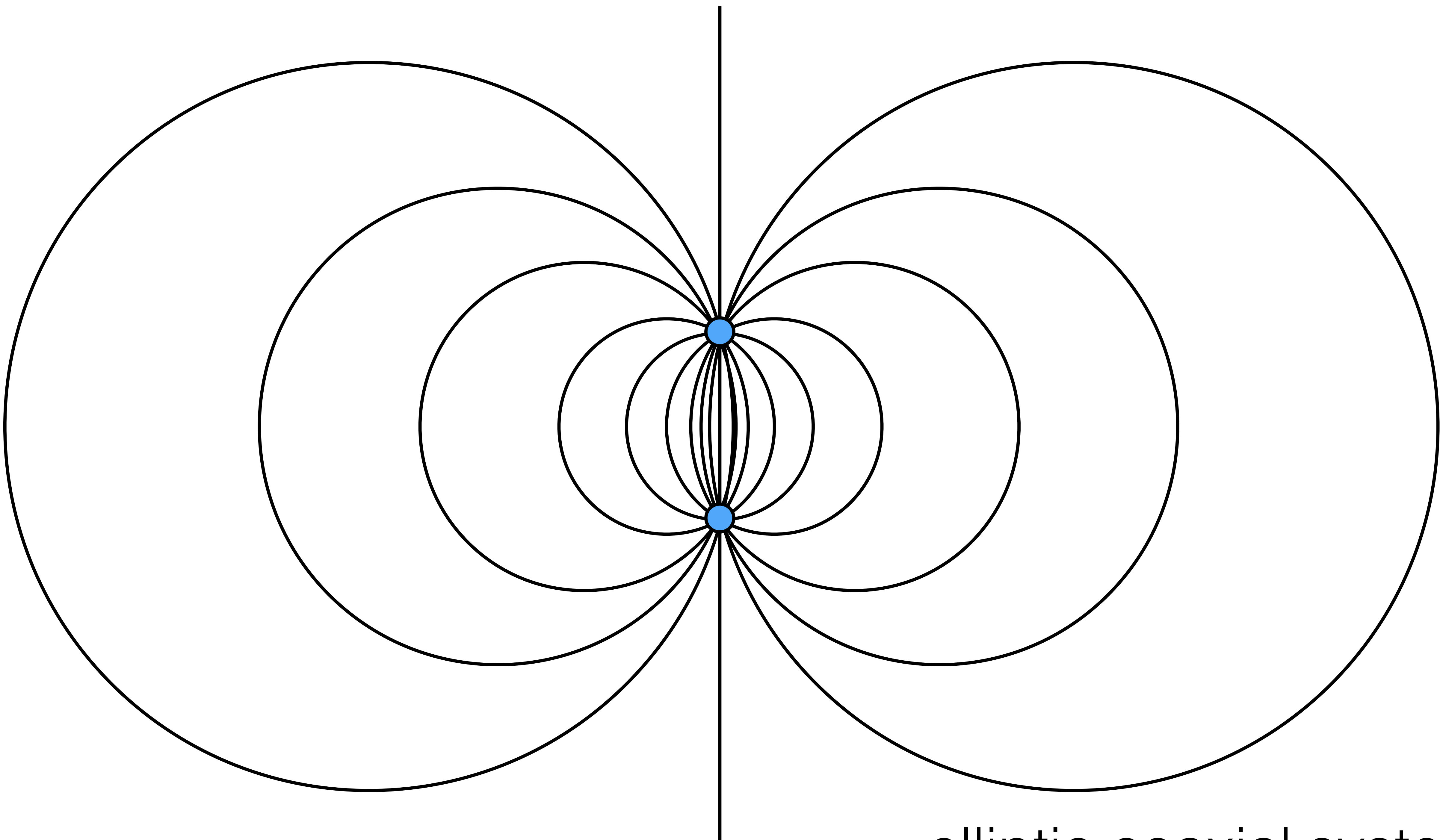
parabolic coaxial system

Two circles defines a family



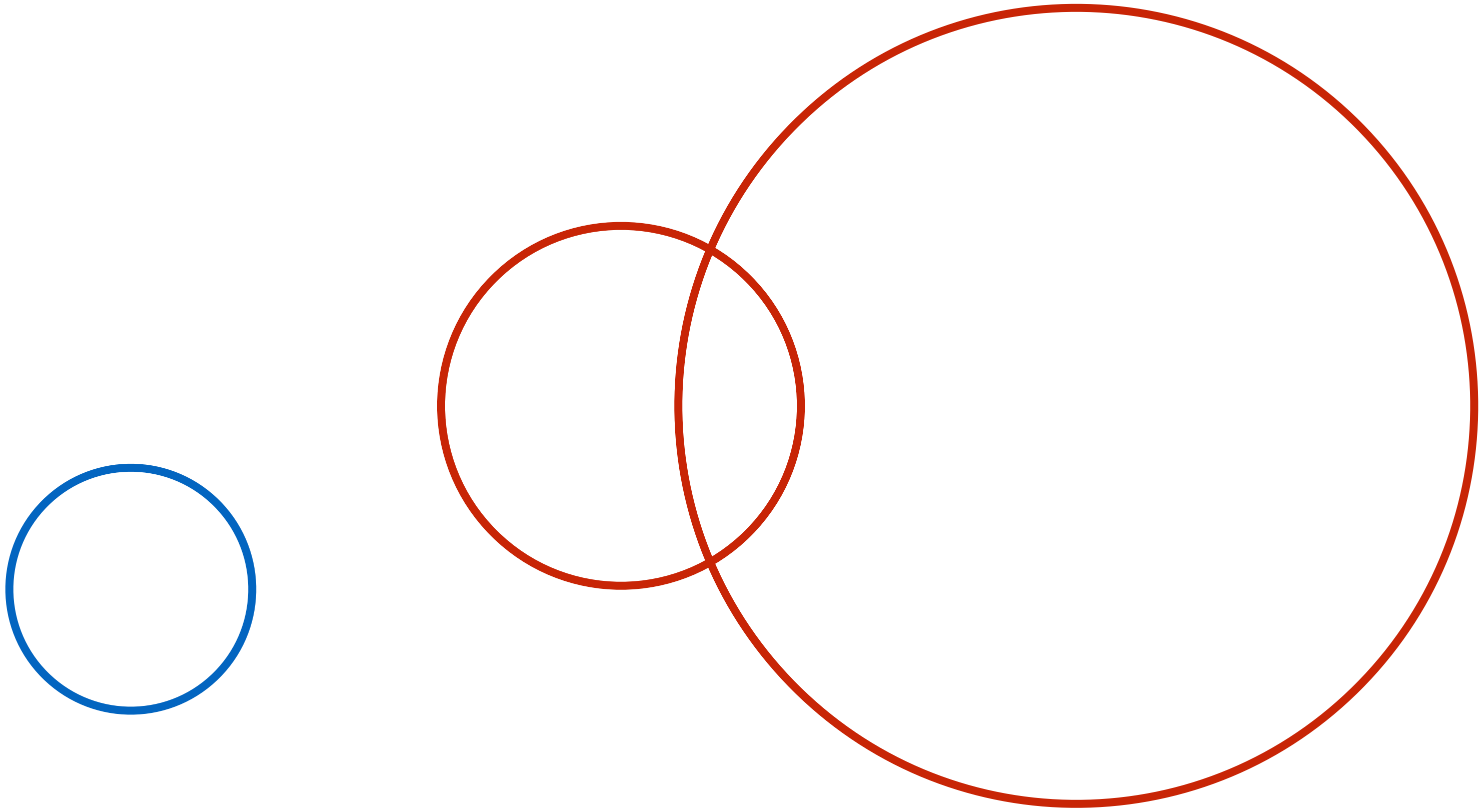
elliptic coaxial system

Two circles defines a family

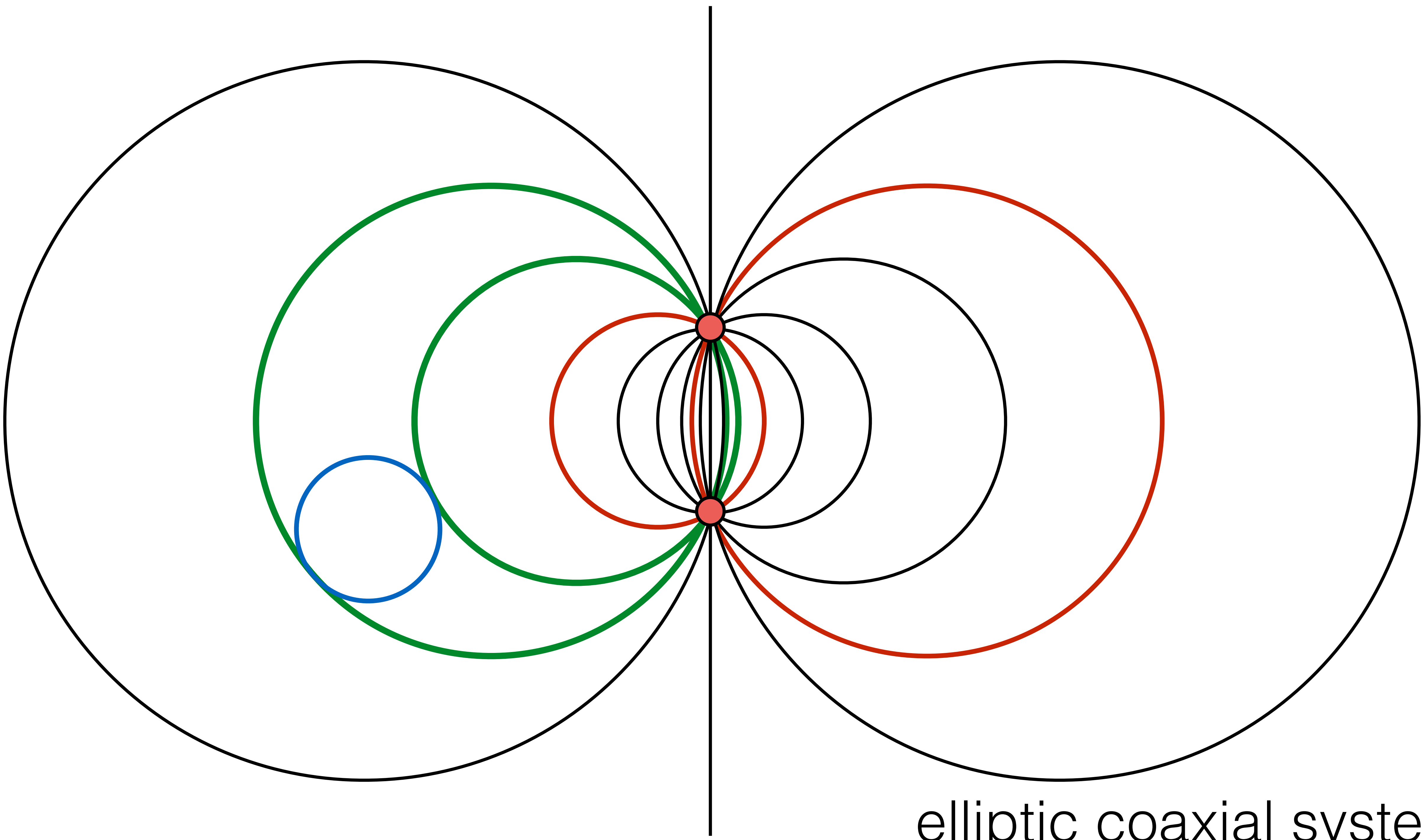


elliptic coaxial system

Flows

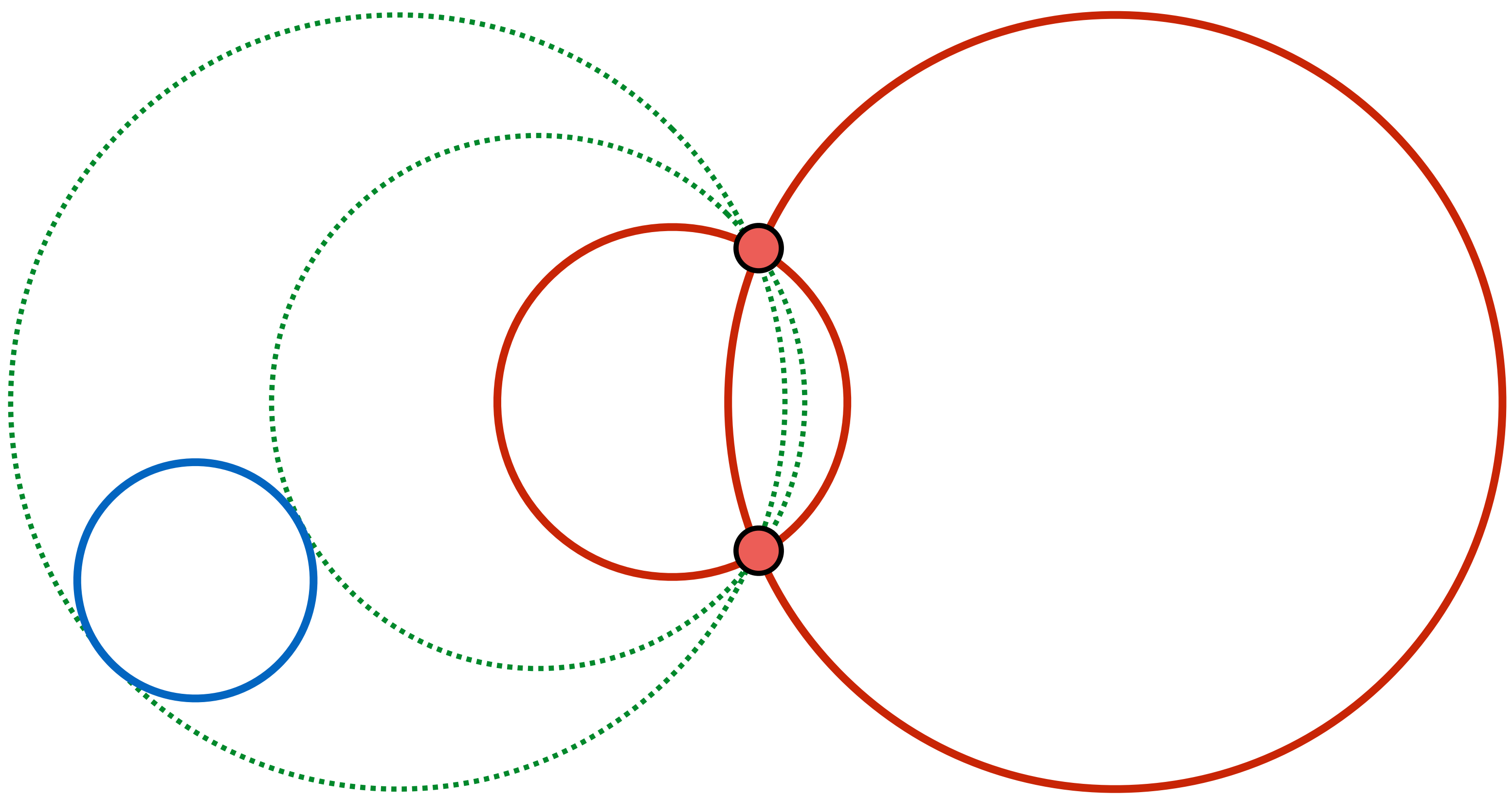


Flows

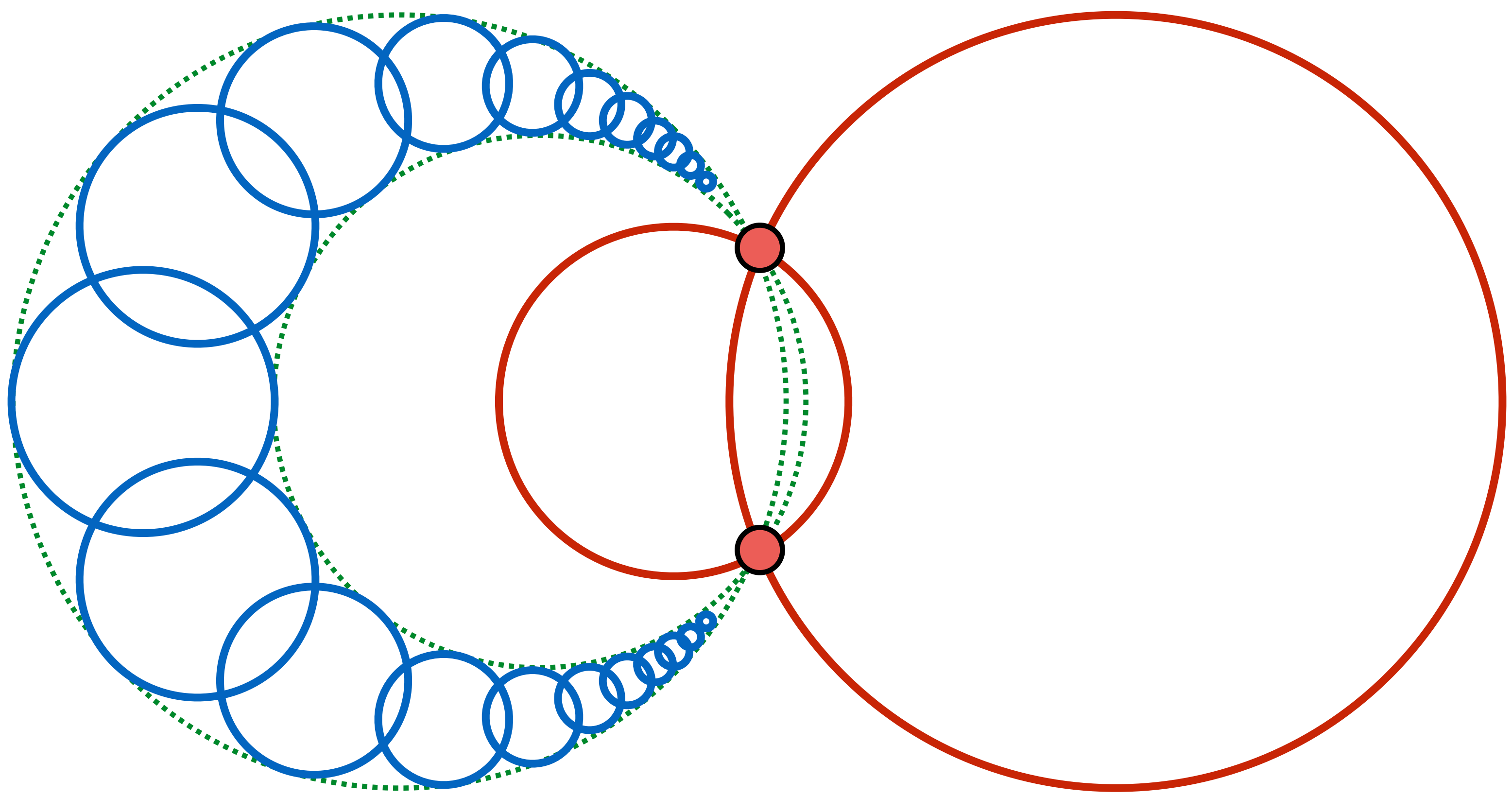


elliptic coaxial system
hyperbolic flow

Flows

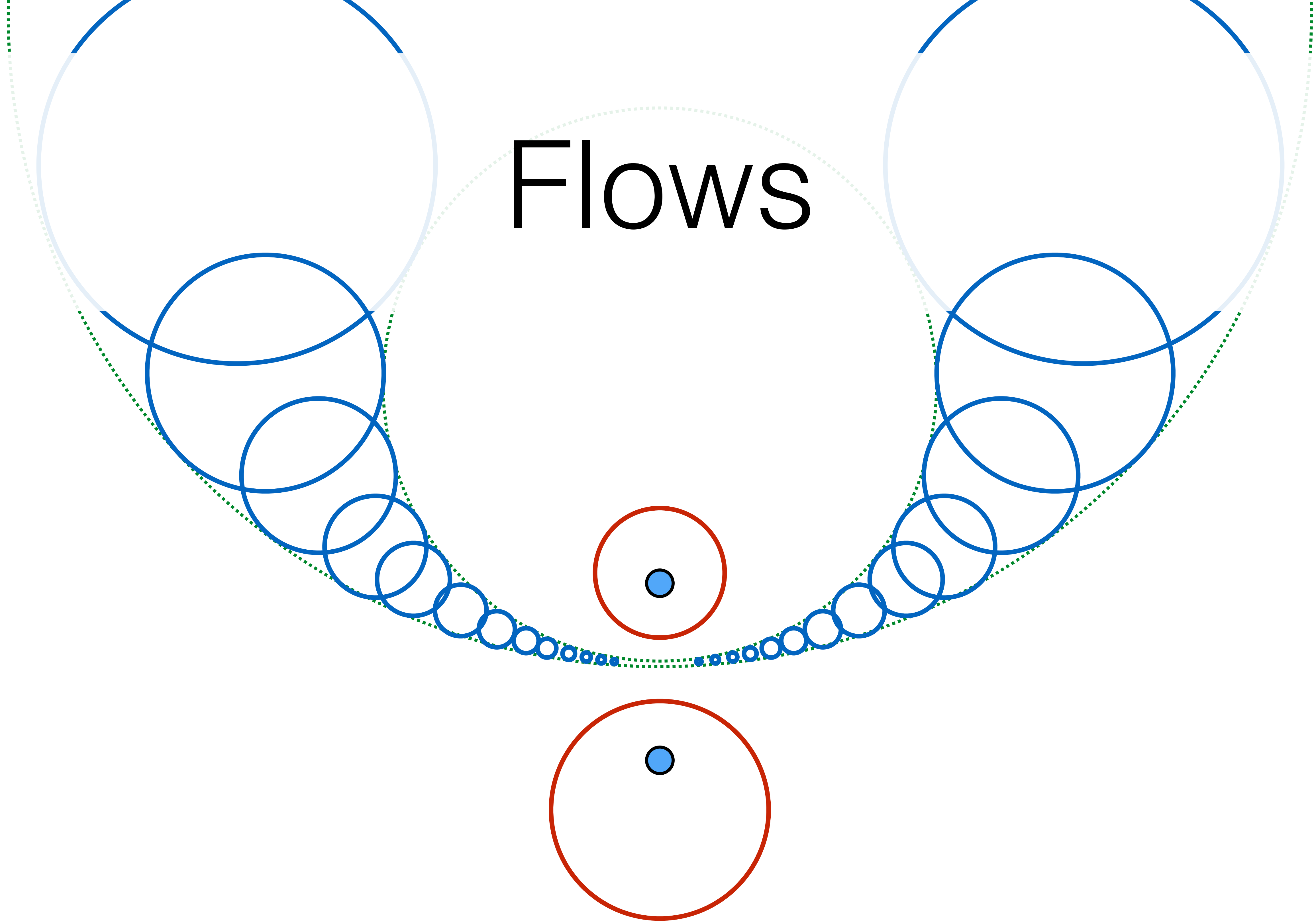


Flows



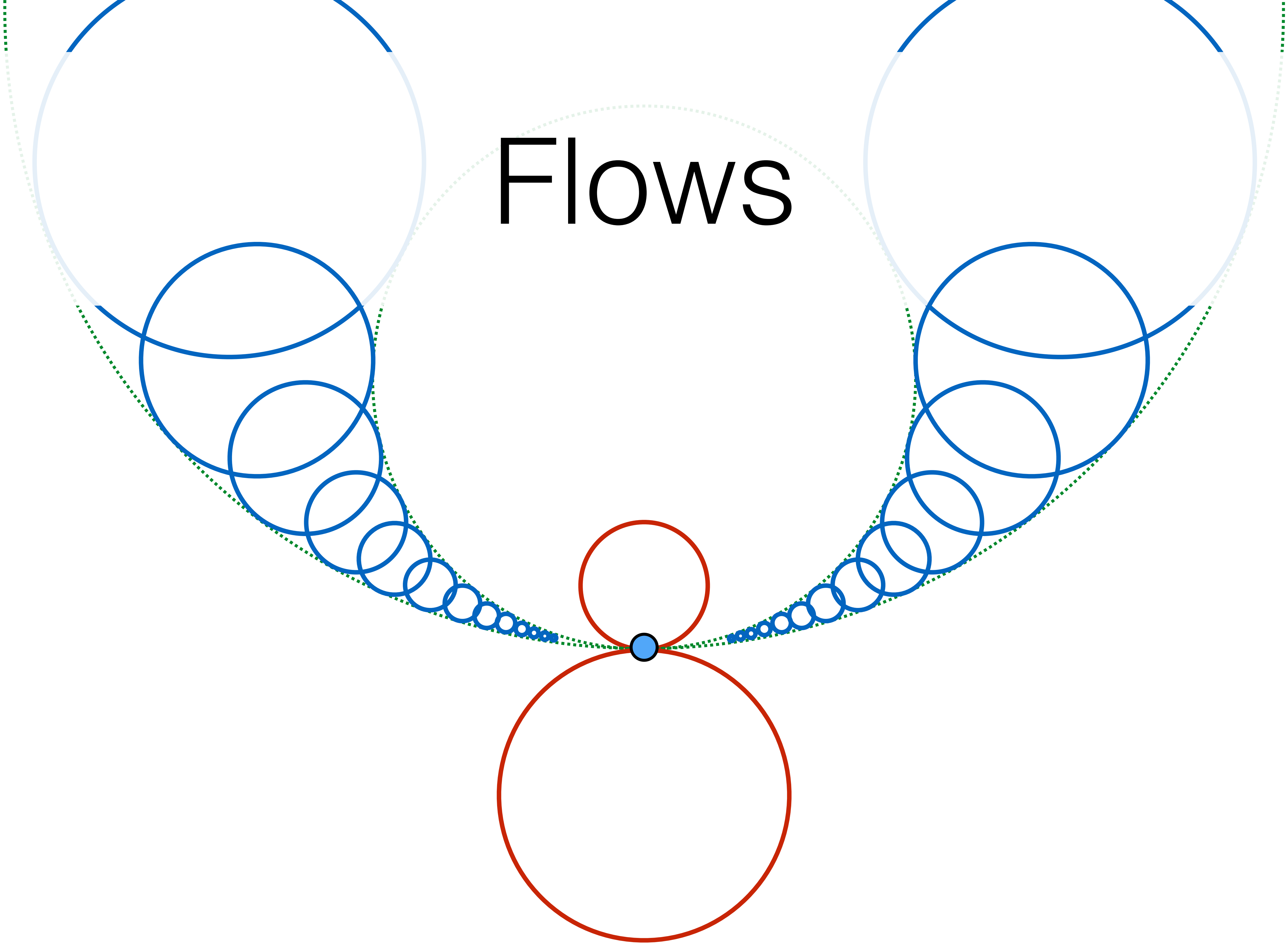
Key Property: Maintains the inversive distance from the blue circle to both red circles.

Flows



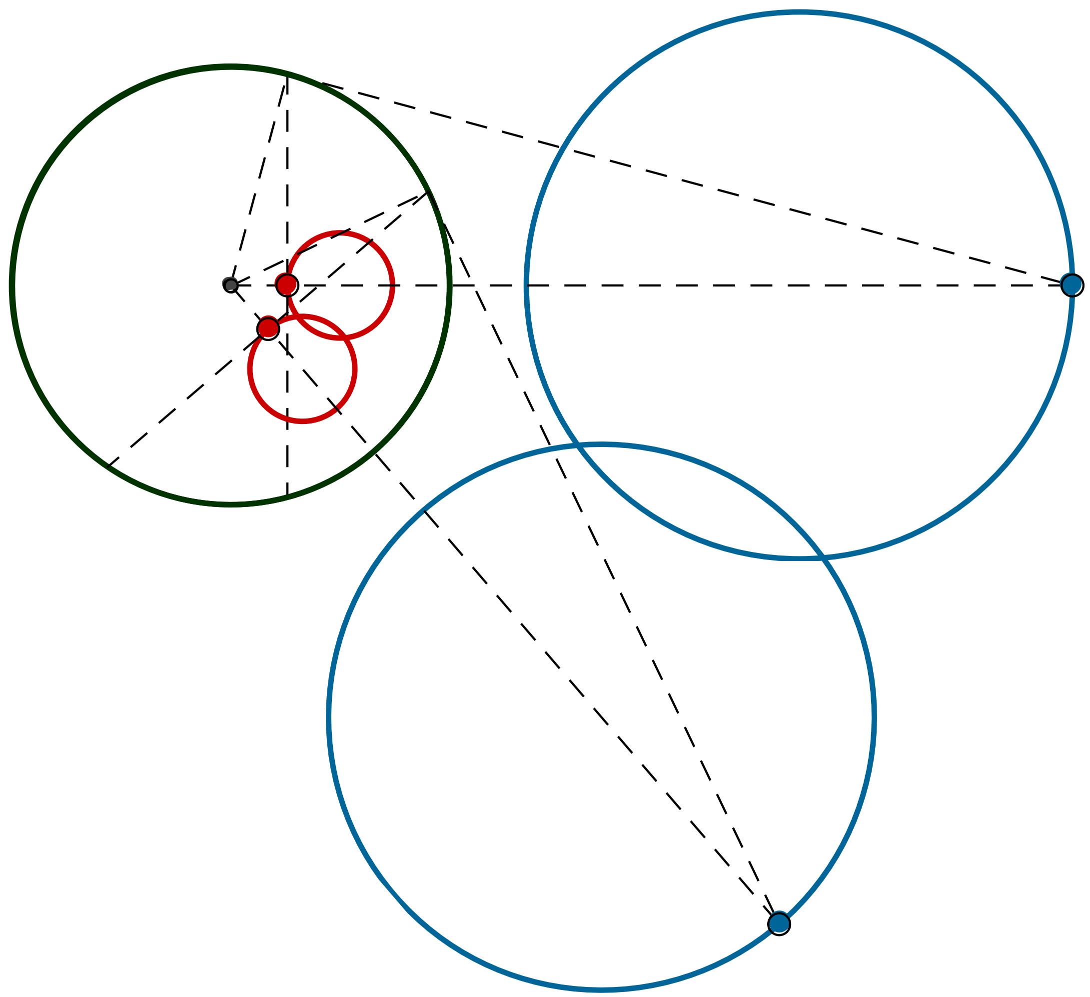
hyperbolic coaxial system
elliptic flow

Flows



parabolic coaxial system
parabolic flow

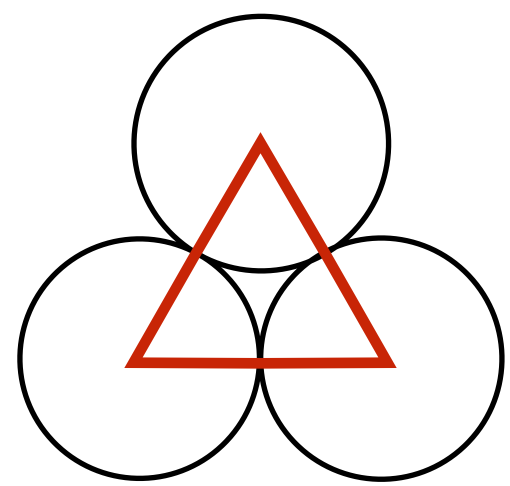
Inversions



Inversions maintain inversive distances (inversive distance between the red circles is the same as the blue pair)

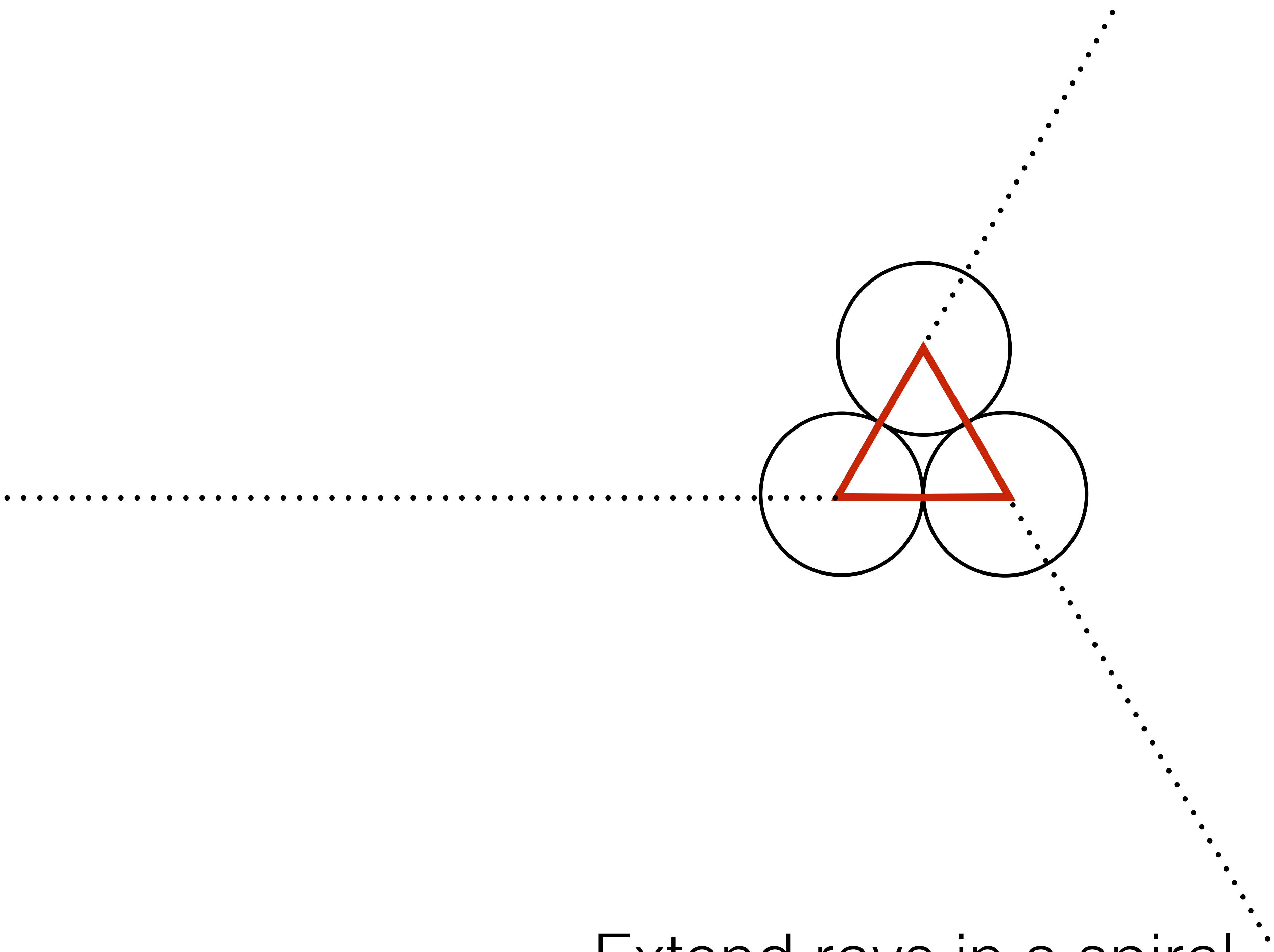
Construction

Ma-Schlenker style Octahedra Construction



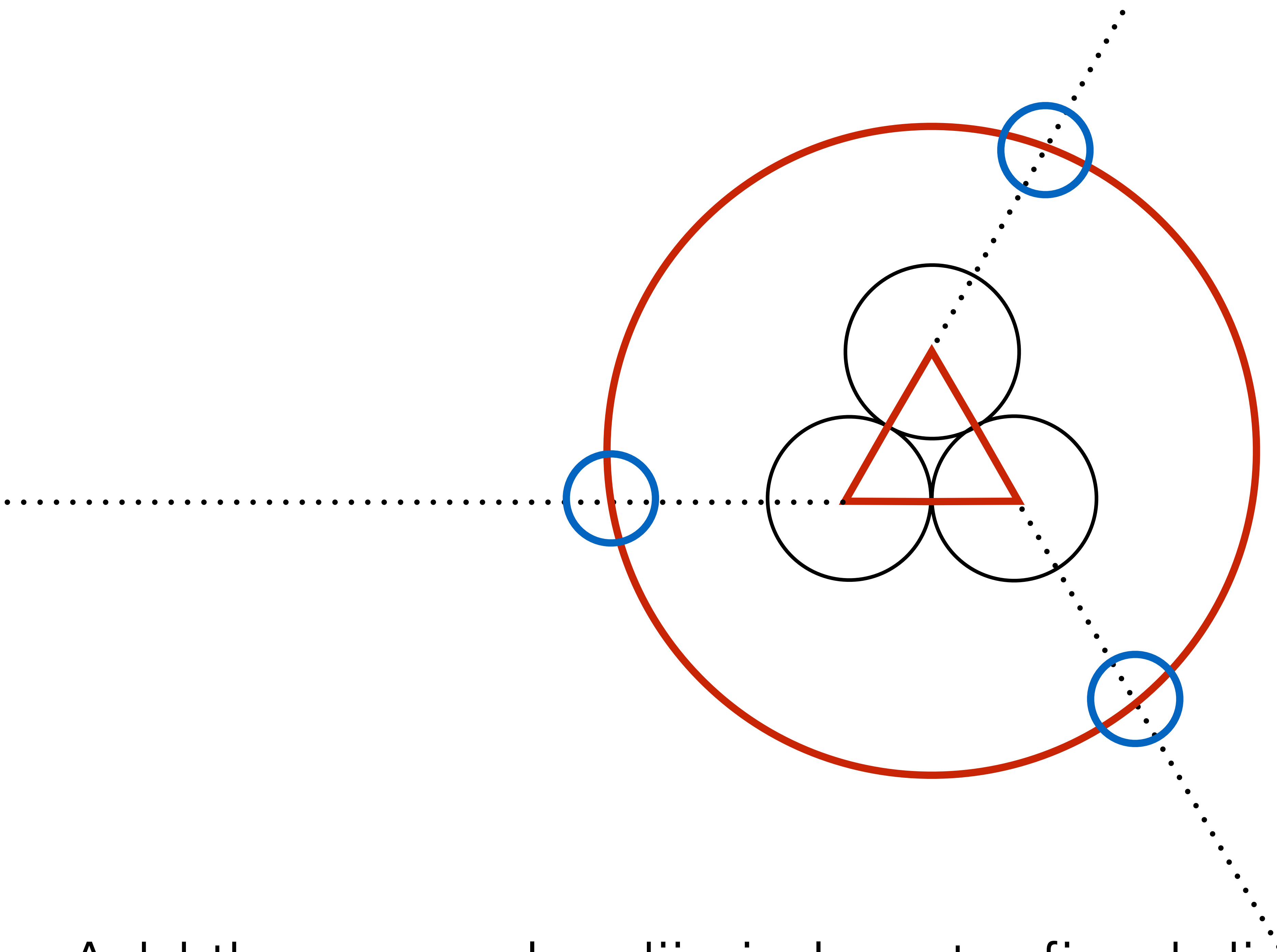
Start with 3 circles with equal radii and centers on an equilateral triangle (not necessarily tangent).

Ma-Schlenker style Octahedra Construction



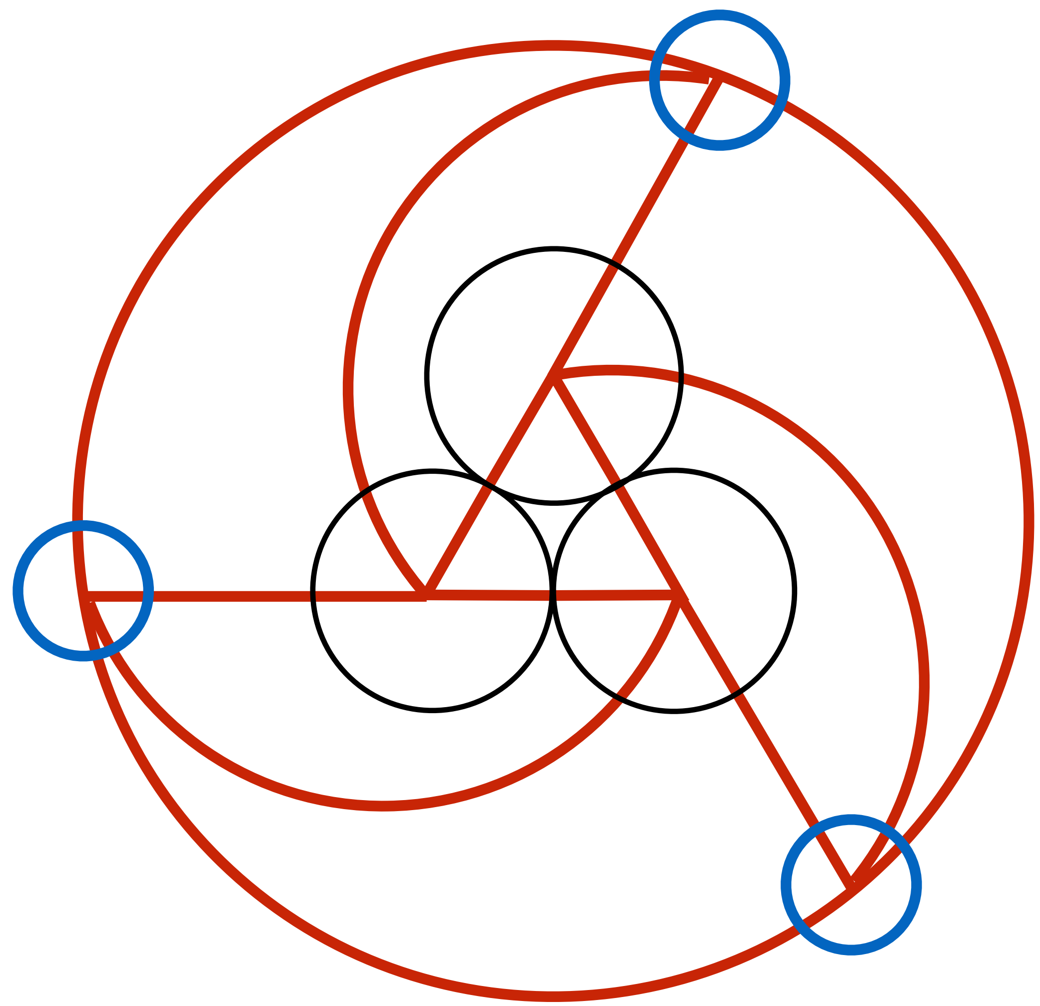
Extend rays in a spiral.

Ma-Schlenker style Octahedra Construction



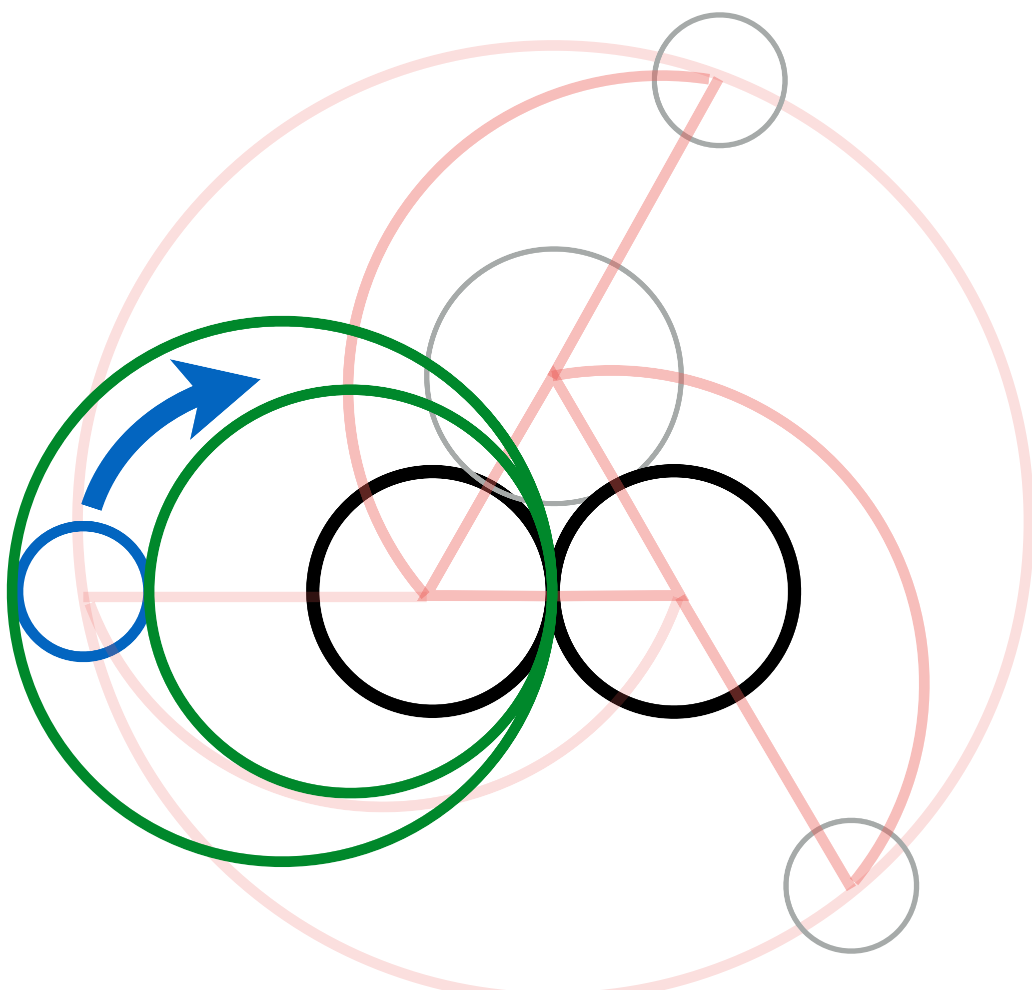
Add three equal radii circles at a fixed distance along rays.

Ma-Schlenker style Octahedra Construction



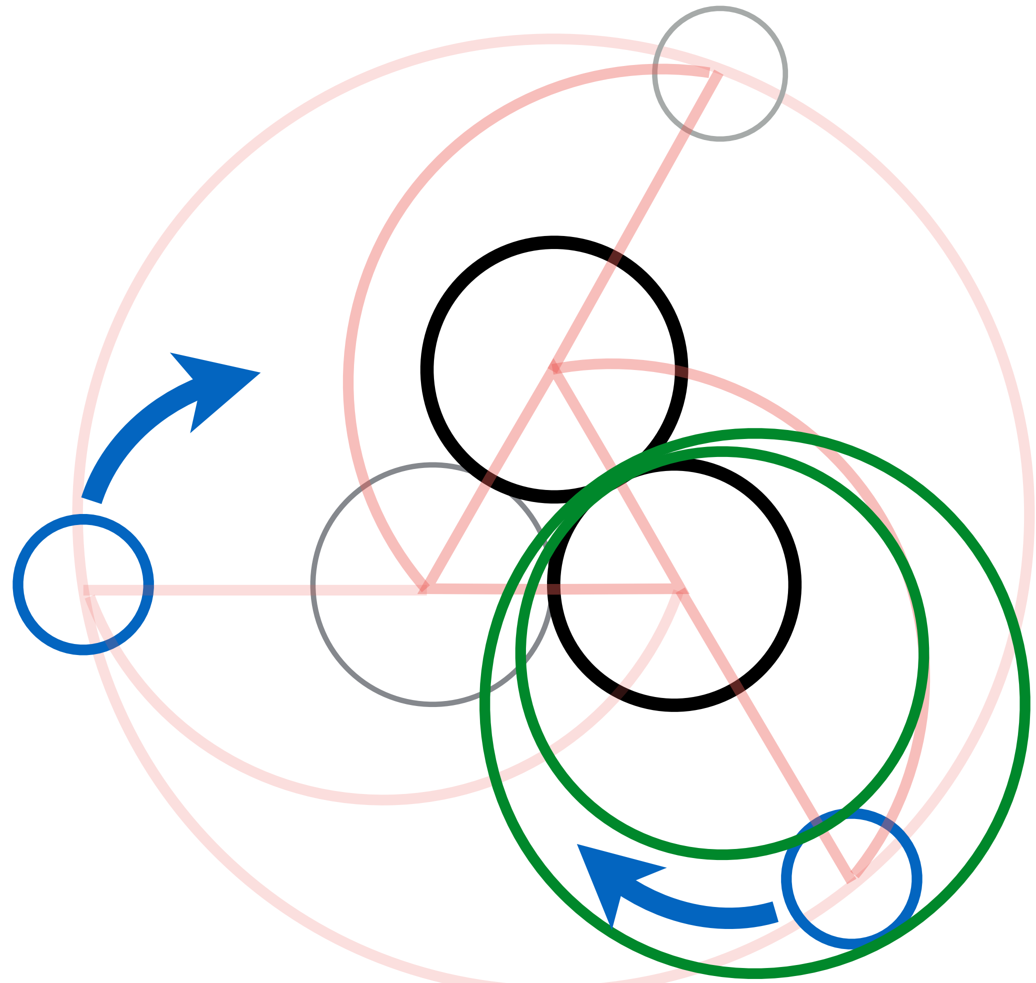
The starting octahedron.

Ma-Schlenker style Octahedra Construction



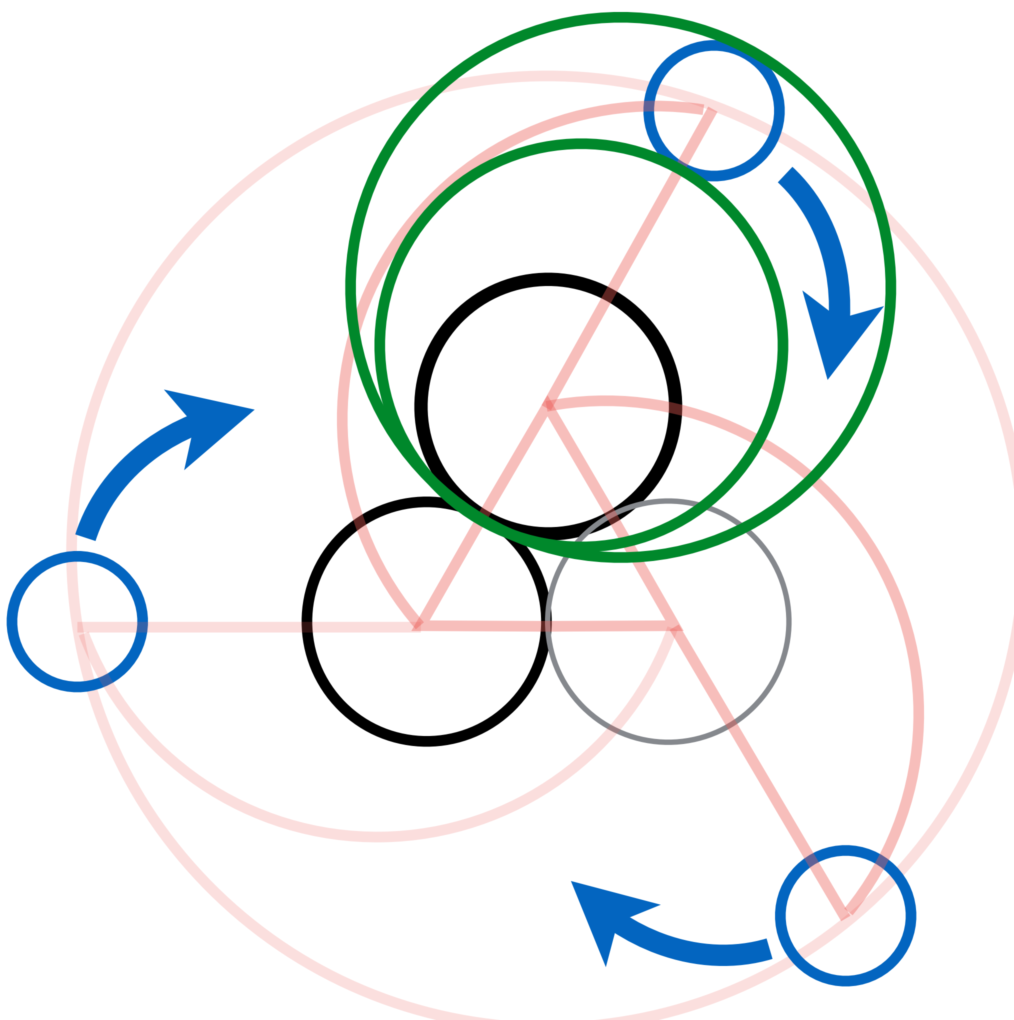
Flow the blue circle.

Ma-Schlenker style Octahedra Construction



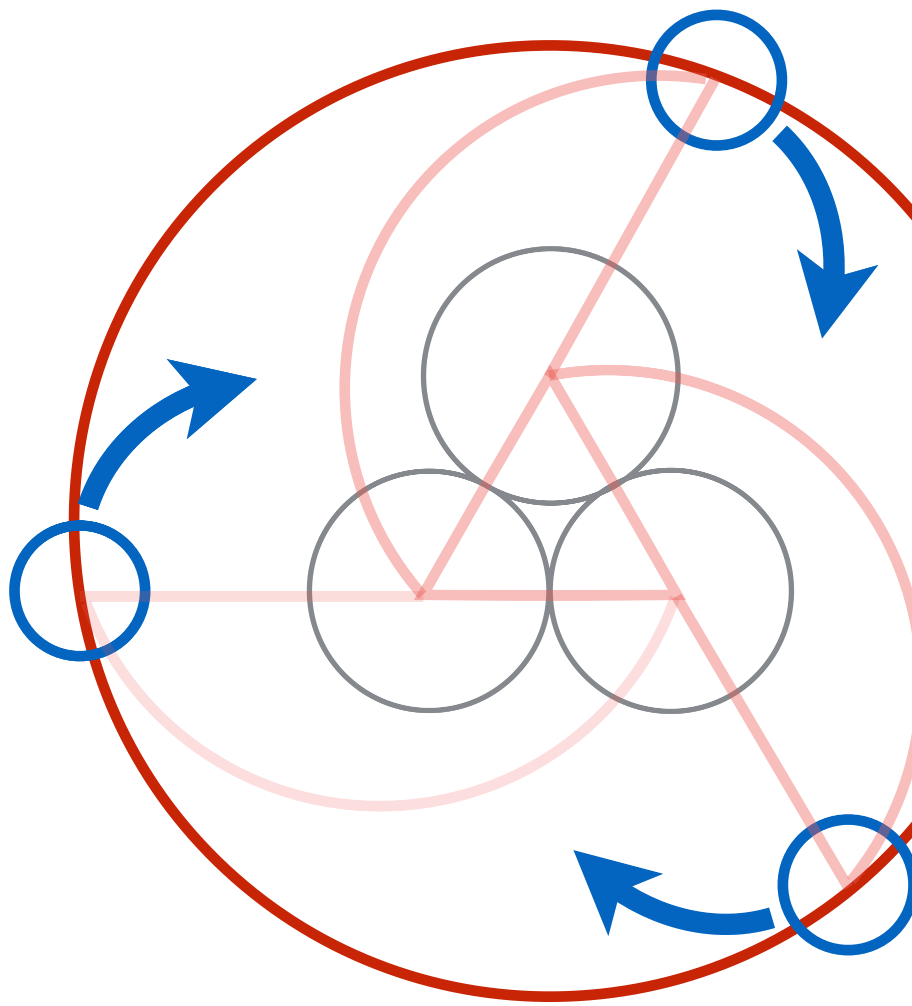
Flow the blue circle.

Ma-Schlenker style Octahedra Construction



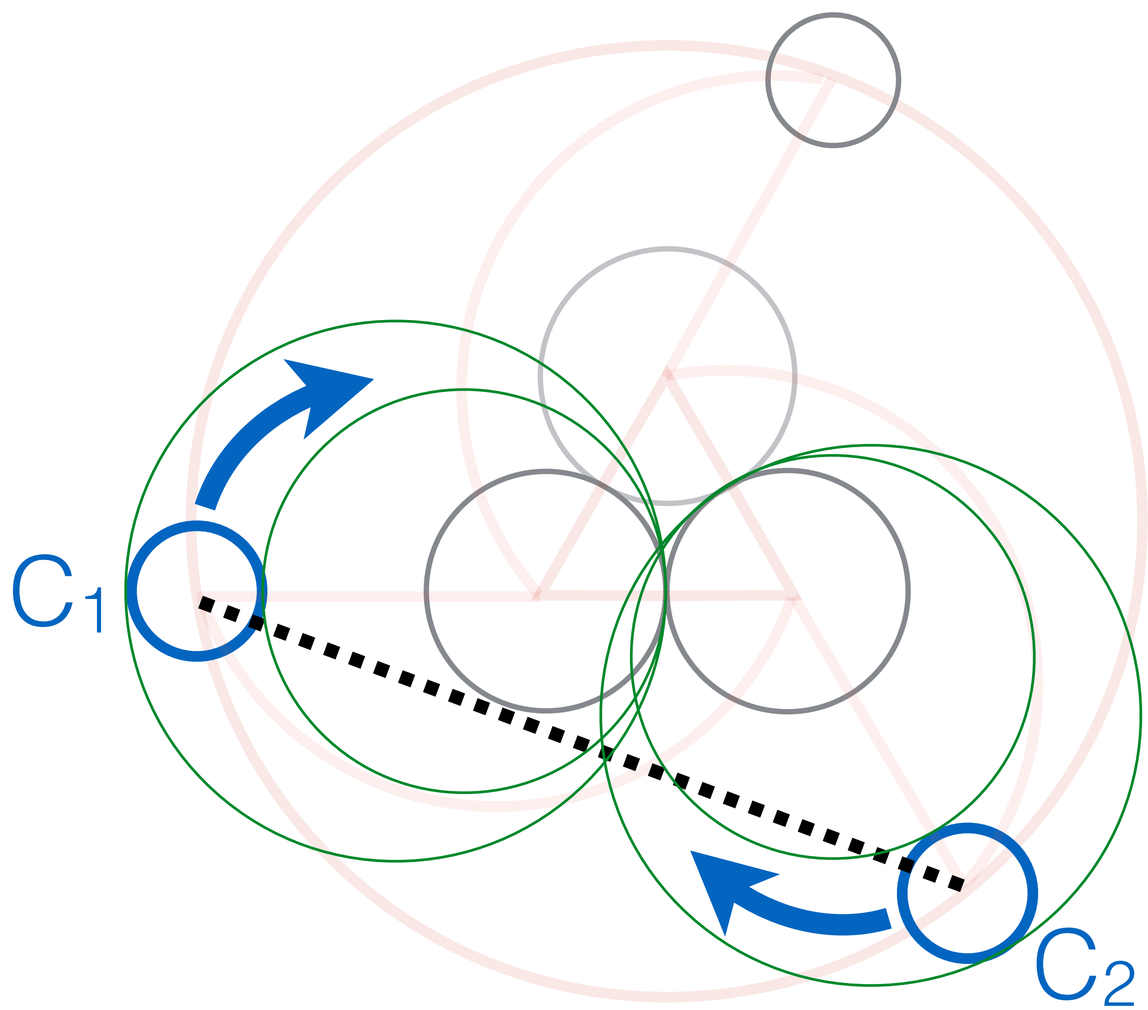
Flow the blue circle.

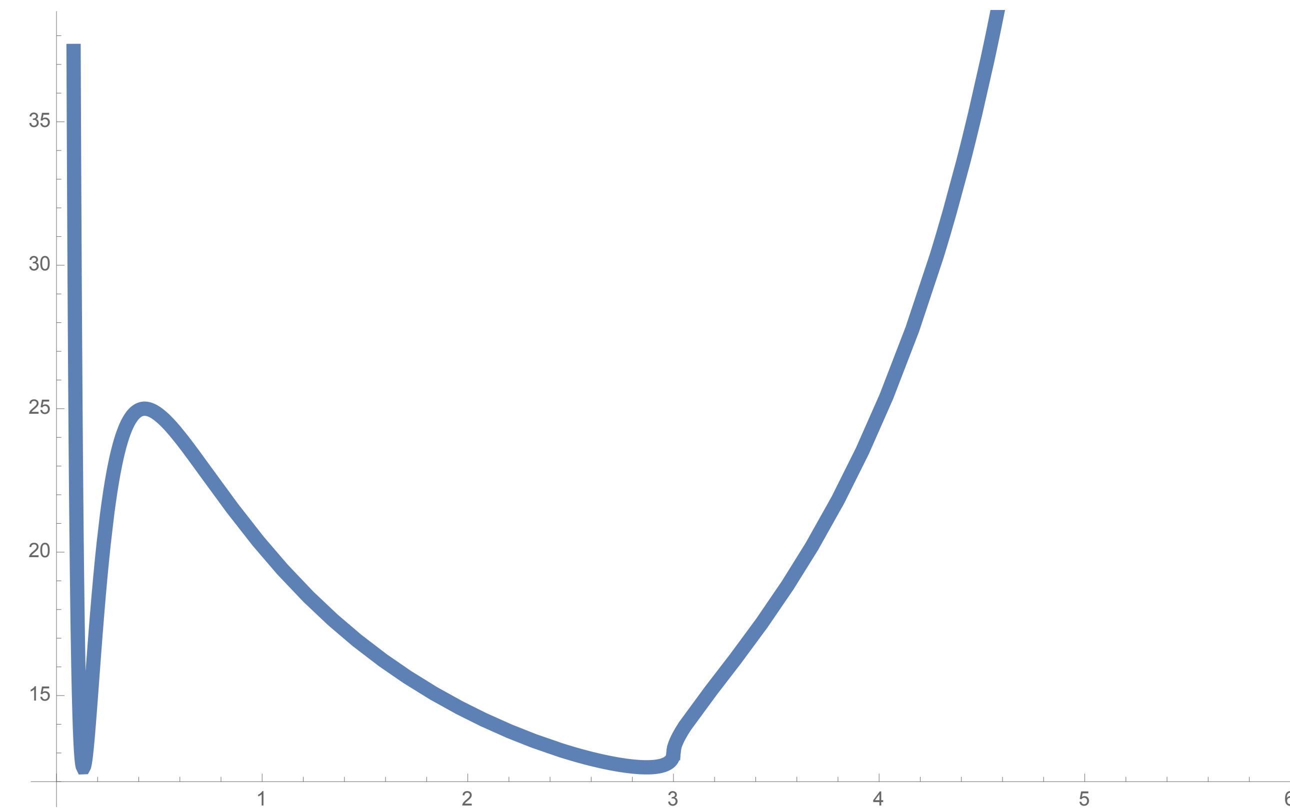
Ma-Schlenker style Octahedra Construction



Only the outer triangle's inversive distances change.
Goal: to find a minimum.

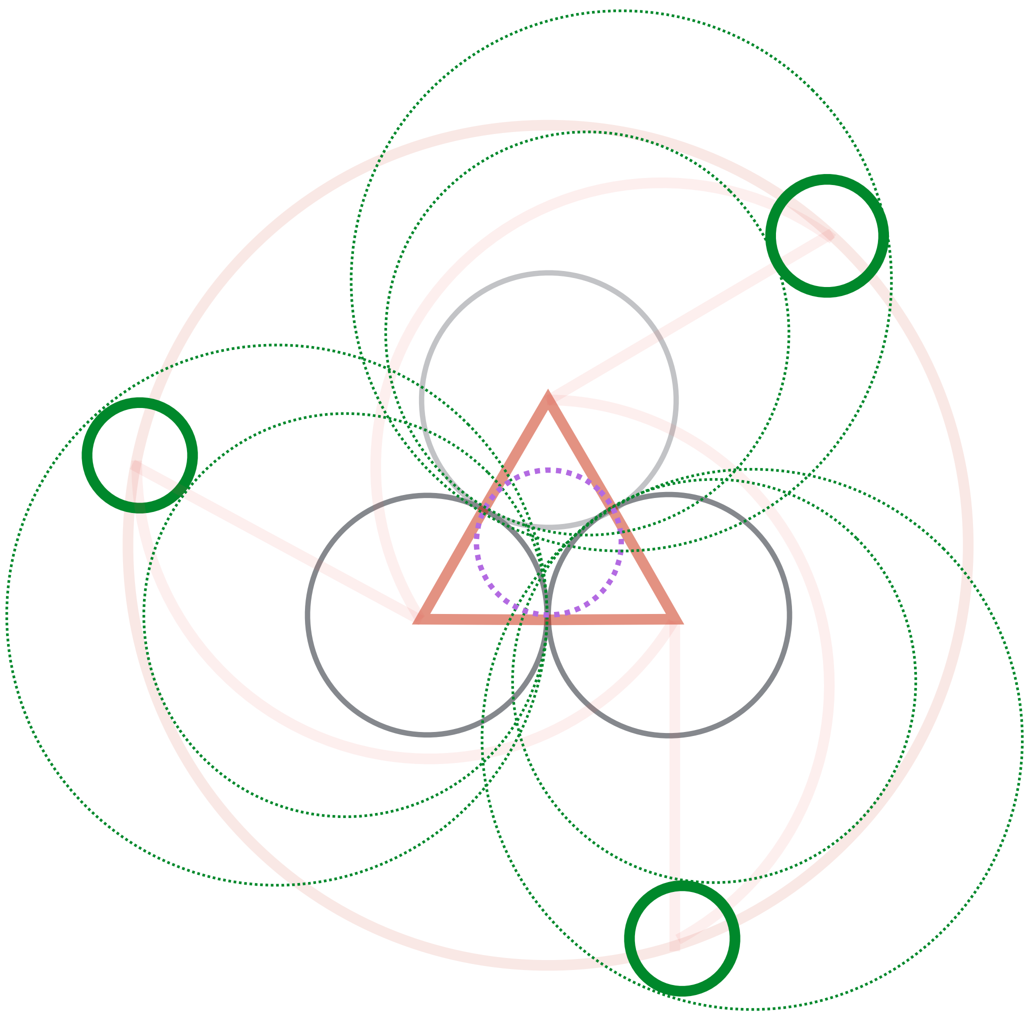
Ma-Schlenker style Octahedra Construction

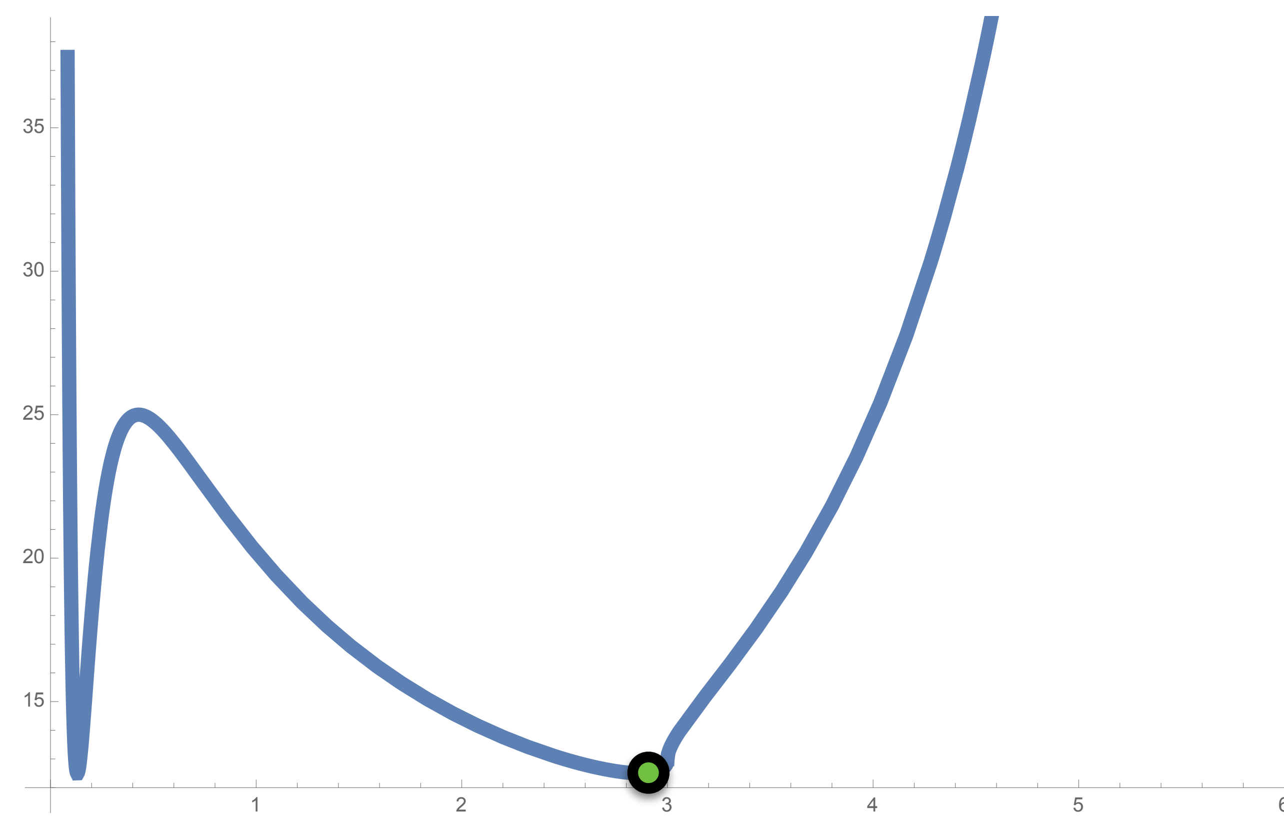




plot of inversive distance between C_1 and C_2 throughout flow

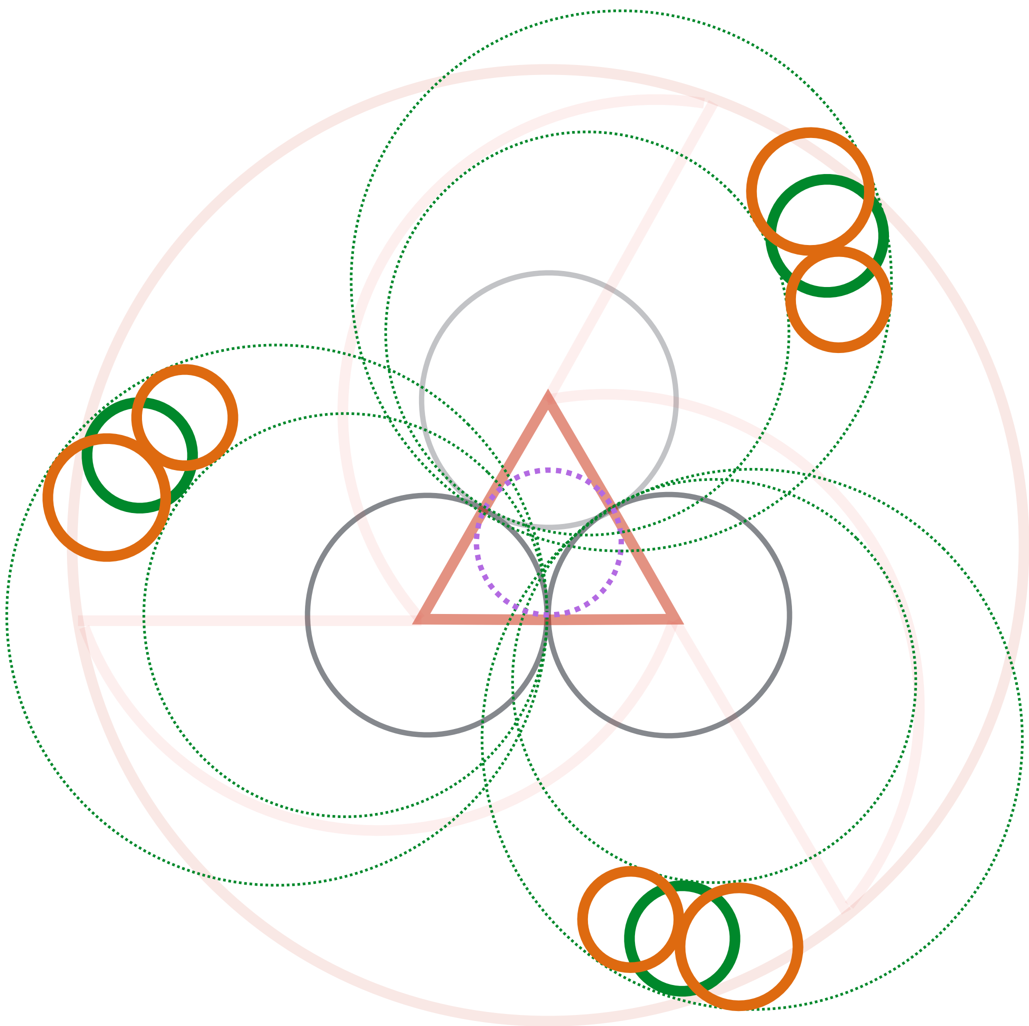
The Critical Octahedron

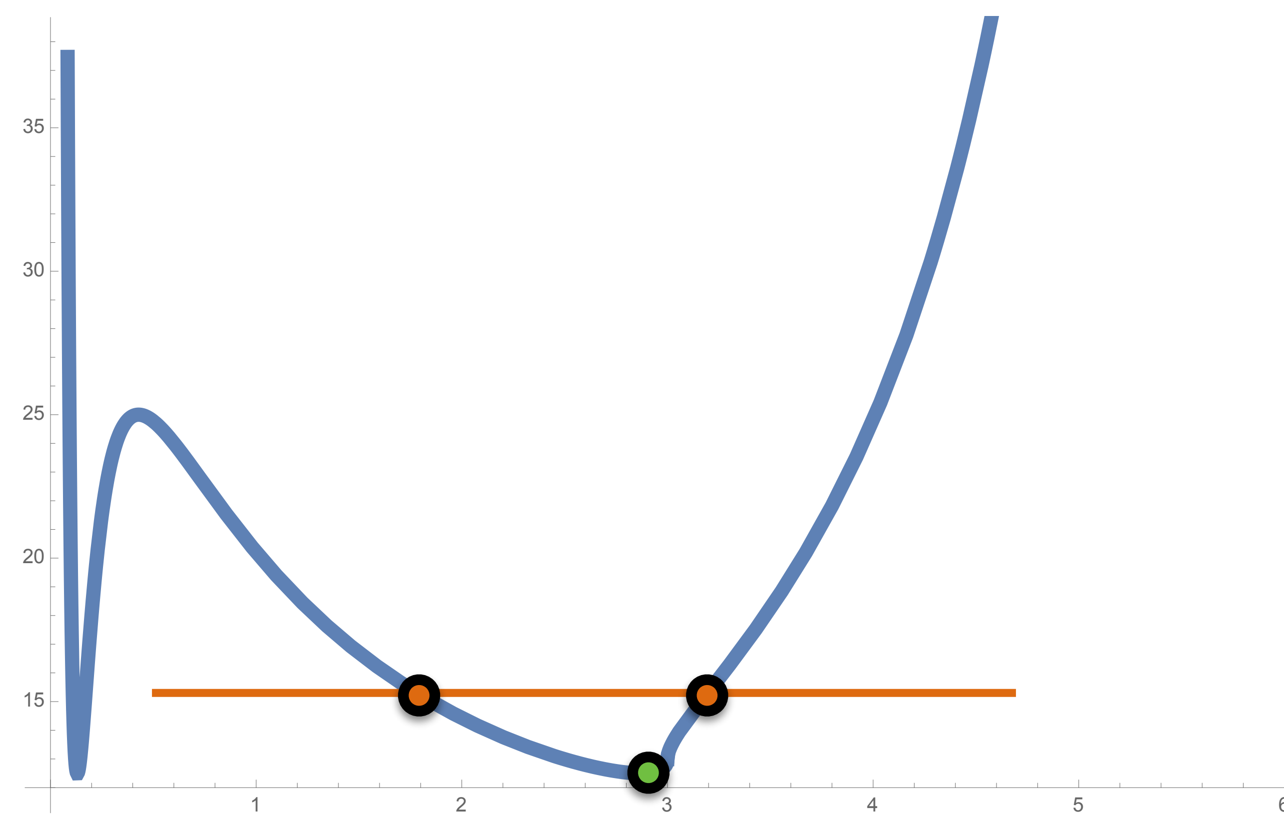




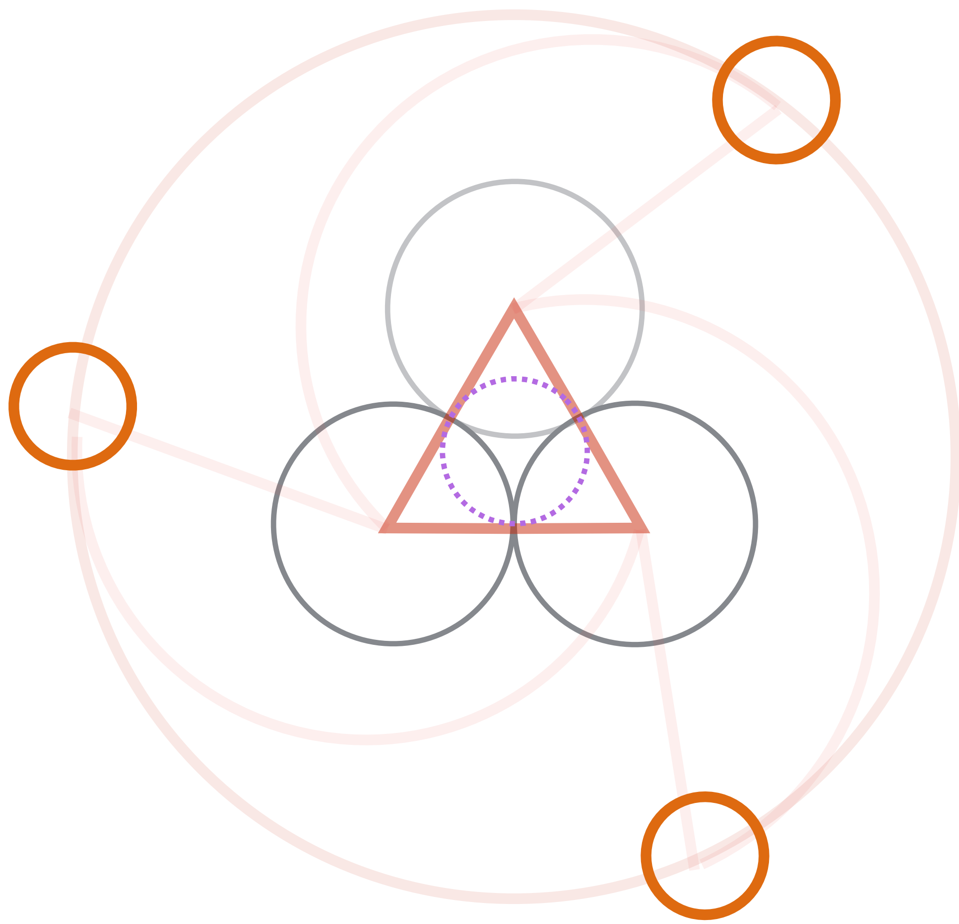
Minimum

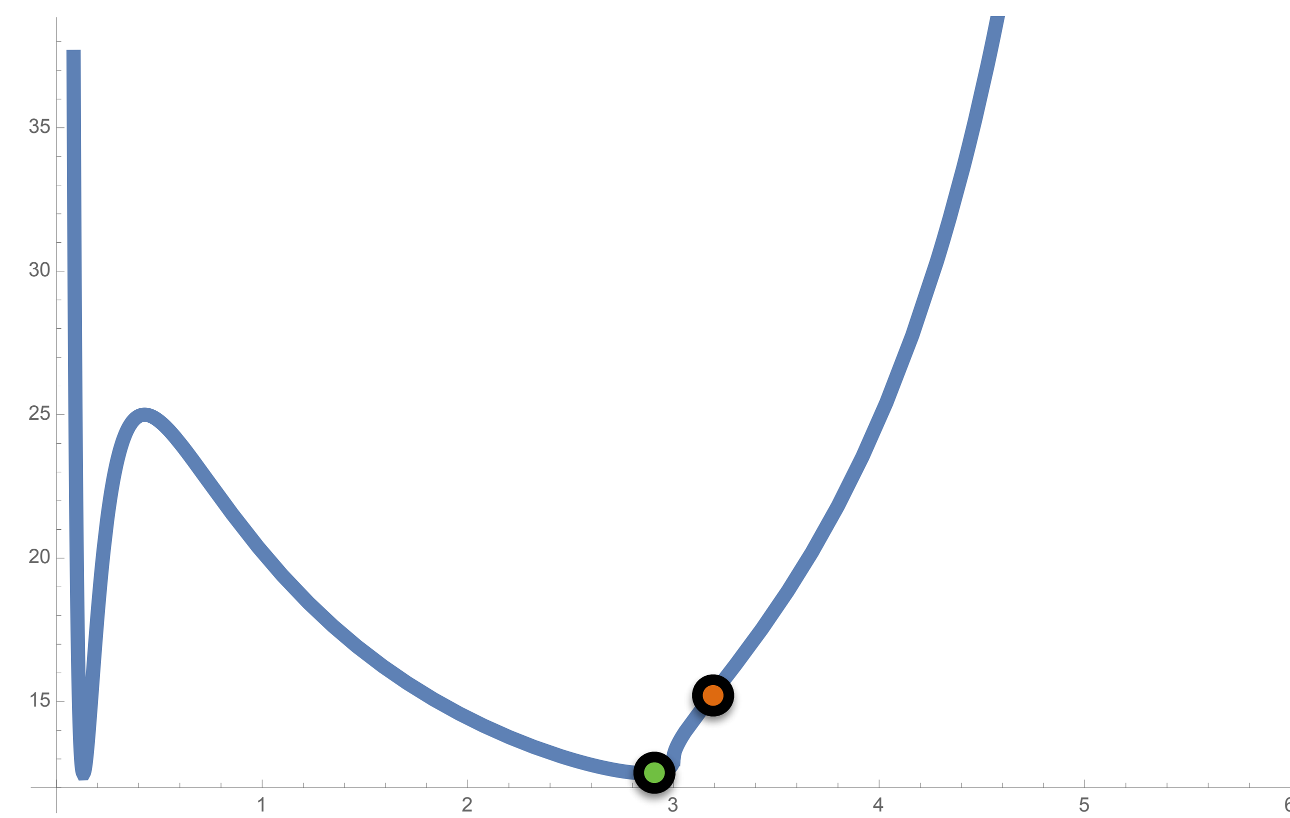
Ma-Schlenker style Octahedra Construction



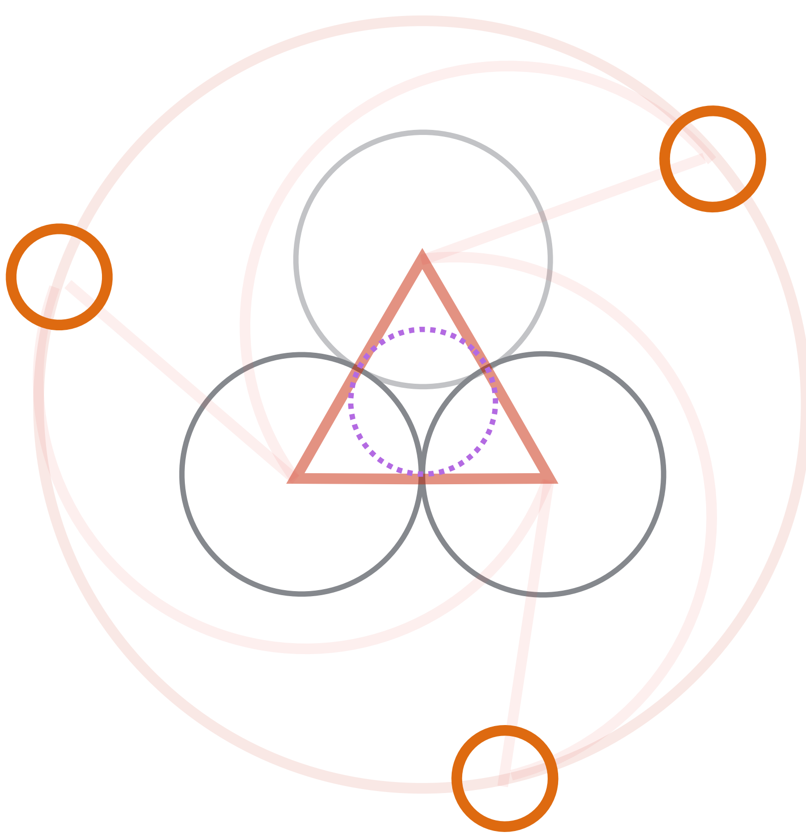


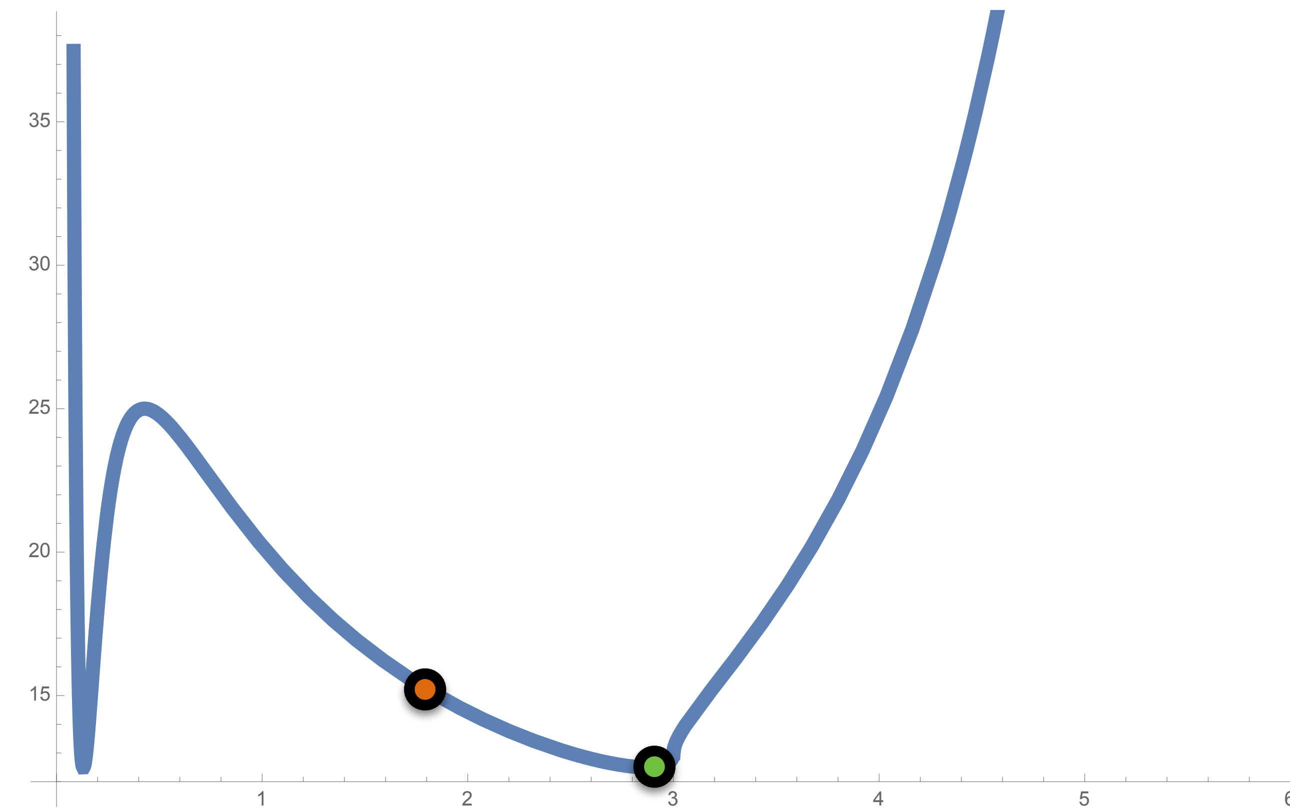
Ma-Schlenker style Octahedra Construction



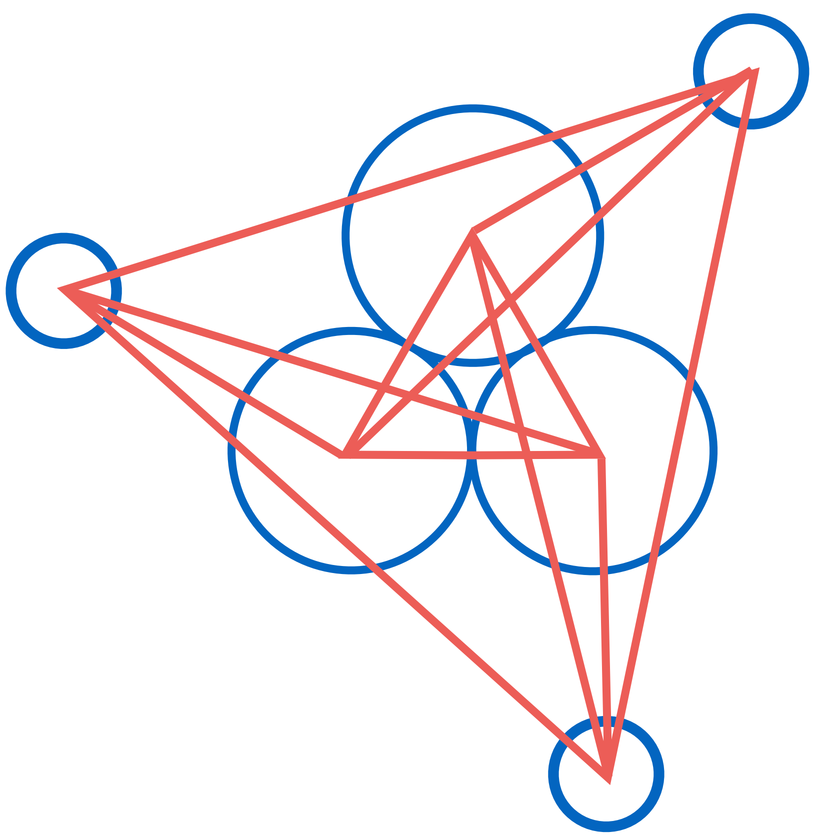


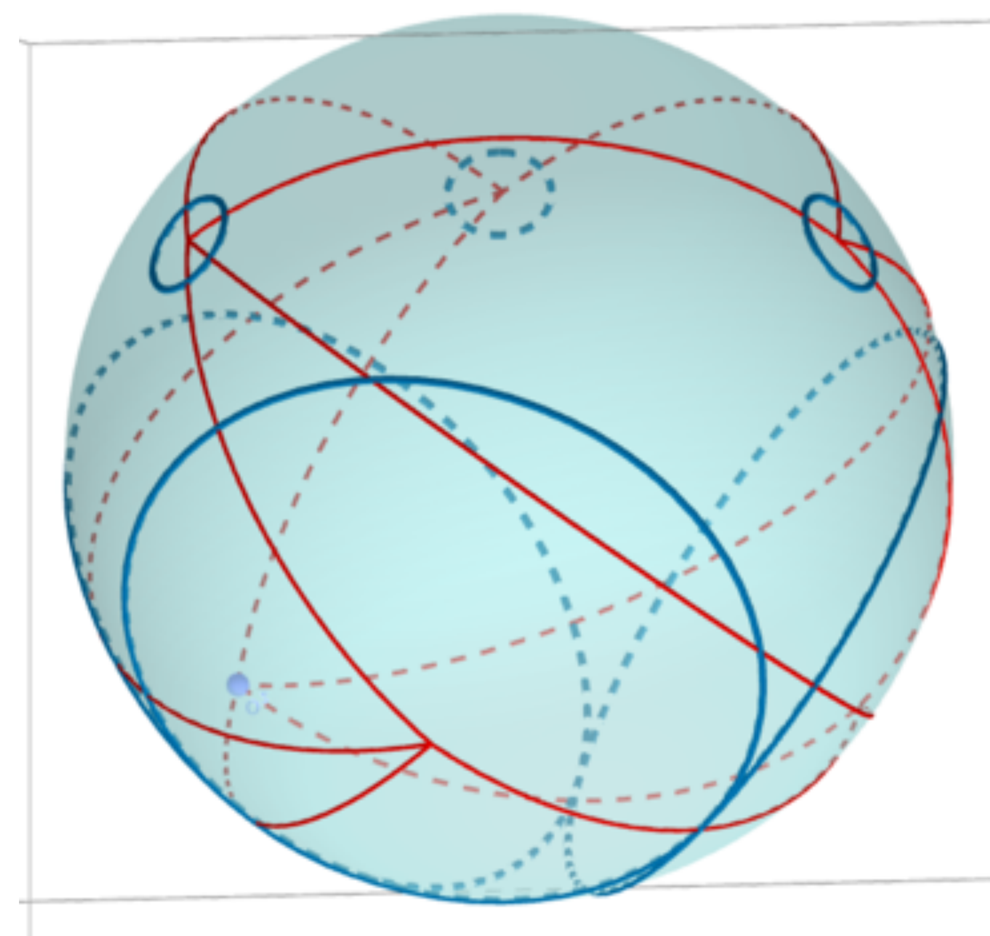
Ma-Schlenker style Octahedra Construction





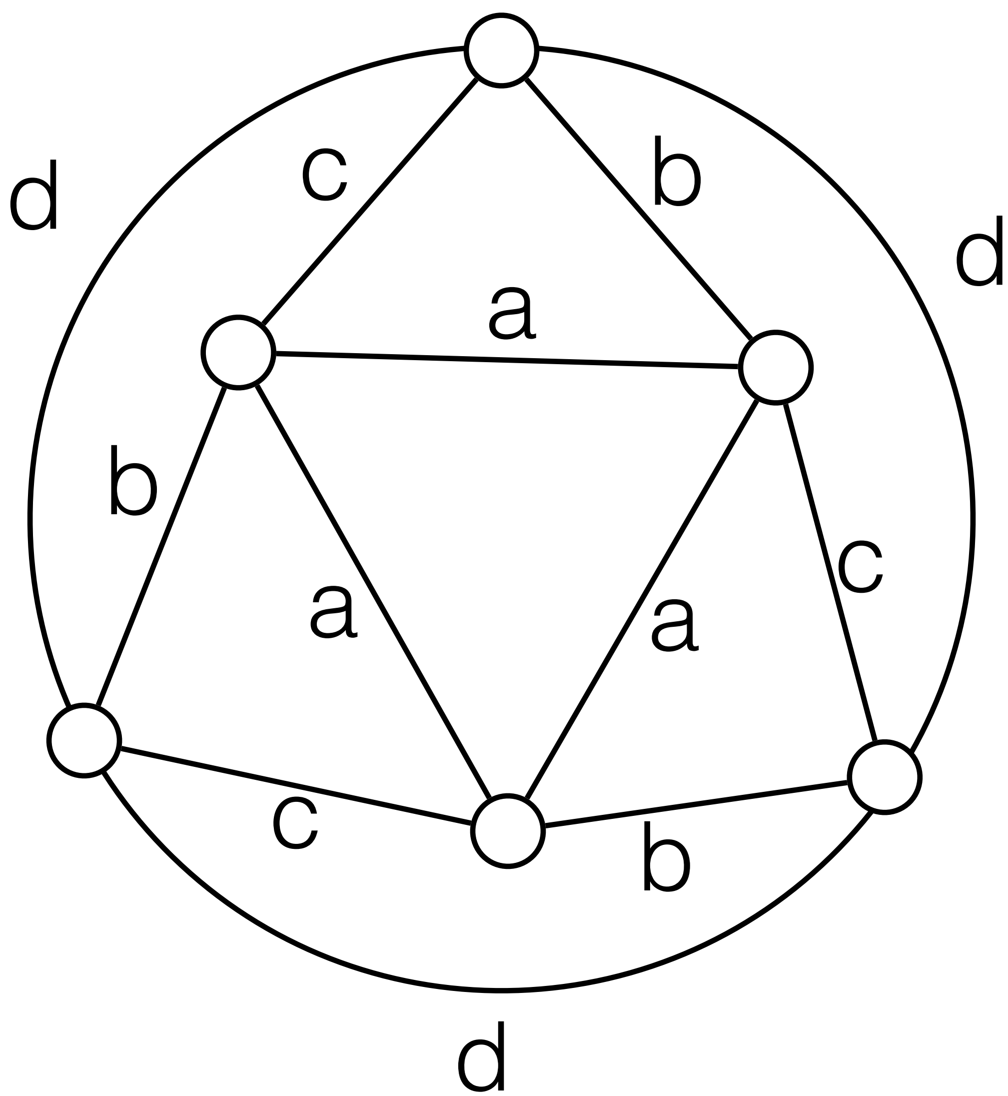
Ma-Schlenker style Octahedra Construction





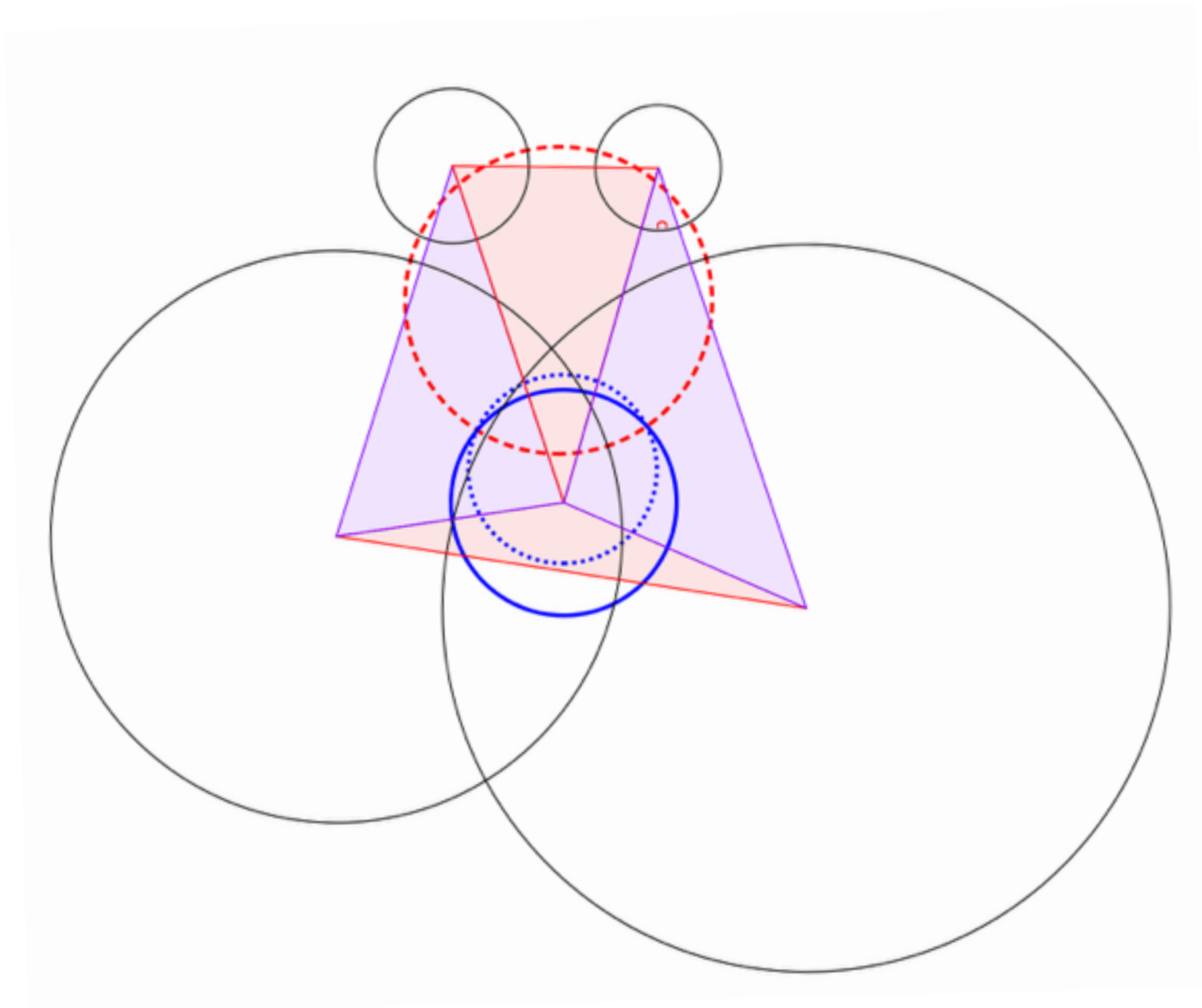
When we stereographically project we may not get a triangulation but by an appropriate Möbius transformation we can always obtain one.

Ma-Schlenker-style Octahedral Packing



- **Theorem**: Any segregated inversive distance circle packing with the graph and distances given by the figure on the left that is sufficiently near the critical octahedron is not unique.

Segregation is needed in the plane



Thank You

Questions?

References

- [Guo] Guo, R., 2011. Local rigidity of inversive distance circle packing. *Transactions of the American Mathematical Society, 363*(9), pp.4757-4776.

- [Luo] Luo, F., 2011. Rigidity of polyhedral surfaces, III. *Geometry & Topology, 15*(4), pp.2299-2319.

- [Ma & Schlenker] Ma, J. and Schlenker, J.M., 2012. Non-rigidity of spherical inversive distance circle packings. *Discrete & Computational Geometry, 47*(3), pp.610-617.