4. Probabilistic Localization

4.1 Introduction

In previous chapters we assumed that the we had access to accurate information about the state of the world. For example, in Chapter 1 we developed a closed-loop controller for moving a robot to a goal by repeatedly comparing the robot's current location to the goal location. This raises the question of how we can know the exact location of the robot. The short answer is that we can't. It is possible to estimate the robot's location by using two general sources of information:

- **Sensors** - There are a wide range of sensors that can provide information about the position of a robot. Cameras may be used to detect landmarks. Depth sensors may be used to estimate the robot's position in a map. Bump sensors may be used to determine when the robot is in contact with an obstacle.
- **Dead reckoning** - Assuming the initial location of a robot is known, it's location at some later time can be estimated by considering the control signals that have been applied. If we send commands telling the robot to move forward at 1m/s for 1s, the robot should end up one meter ahead of the initial location.

There is inherent uncertainty associated with each of these sources of information. No sensor is perfect. Dead reckoning is never perfectly reliable. The goal of this chapter is to introduce a probabilistic framework that will make it possible to represent and reason about this uncertainty.

For the sake of concreteness, this chapter will focus on probabilistic representations of a robot's location, but the same mathematical tools are useful for representing any uncertain information.

4.2 Discrete Probability Distributions

A **discrete random variable**, usually expressed as an upper-case letter such as X, is a variable that can take on a fixed number of possible values. A **probability distribution**, also called a **probability mass function**, is a function that maps from each possible value of the random variable to its probability.

As a simple example, consider a Boolean random variable H that describes the possible outcomes of flipping a coin. In this case, the possible values for H are $True$ indicating that the coin landed on heads or $False$ indicating that the coin landed on tails. The probability mass function for a fair coin is then

$$P(H = True) = .5$$
$$P(H = False) = .5$$

In the case of Boolean variables we often use the more concise convention of indicating an assignment of true using a lower case variable, so that $P(H = True)$ could be expressed as $P(h)$ and $P(H = False)$ could be expressed as $P(\neg h)$.

A valid probability mass function must satisfy the following conditions:

- Probabilities must not be negative or greater than one:

  $$0 \leq P(X = x_i) \leq 1 \text{ for all } x_i$$

- The probability mass function must sum to one:

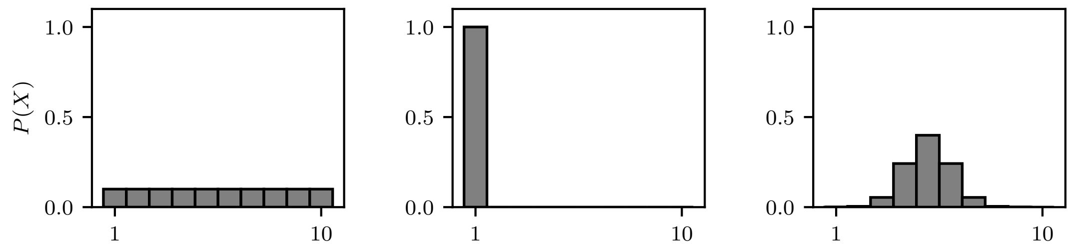
  $$\sum_{x_i} P(X = x_i) = 1$$

Discrete random variables need not be Boolean-valued. In particular, a random variable may be used to describe our uncertainty about the location of a robot, with each possible robot location corresponding to a value for the random variable, and the probability distribution describing our beliefs about which location is correct. For example, consider the problem of tracking the location of a robot in a one-dimensional world (perhaps our autonomous locomotive from Chapter 1). Figure 4.1 illustrates the idea. The horizontal location of each bar corresponds to a discretized value for the position variable, while the height of the bars correspond to the value of the probability mass function for that position. Notice that the height of all bars must always sum to one, since we know that the robot must be located at some location. The same information may also be presented in tabular form as illustrated in Figure 4.2.

In a more realistic scenario, the values for our random variable could be entries in a grid corresponding to possible locations in a two-dimensional workspace. This idea is illustrated in Figure 4.3.

4.2.1 Joint Probabilities

Probabilistic models of complex systems generally involve multiple, interacting, random variables. The multivariate generalization of the probability distribution is the **joint probability distribution**. A joint probability distribution maps from every possible outcome of all variables to the probability for that set of assignments. For example, In the case of our 1-d robot above, we can introduce a second random



(a) (b) (c)

Figure 4.1: Sample histogram probability distributions describing our belief about the location of a robot in a one-dimensional environment. (a) Uniform distribution representing a complete lack of knowledge about the location of the robot. (b) Certain knowledge that the robot is in location 1. (c) Belief that the robot is most likely to be in location 5, but may be in nearby locations to the left or right.

X	P(X)
1	.1
2	.1
3	.1
4	.1
5	.1
6	.1
7	.1
8	.1
9	.1
10	.1

(a)

X	P(X)
1	1.0
2	0
3	0
4	0
5	0
6	0
7	0
8	0
9	0
10	0

(b)

X	P(X)
1	0
2	.004
3	.054
4	.242
5	.400
6	.242
7	.054
8	.004
9	0
10	0

(c)

Figure 4.2: Tabular representations of the probability distributions illustrated in Figure 4.1. Notice that the rows sum to one in each table.

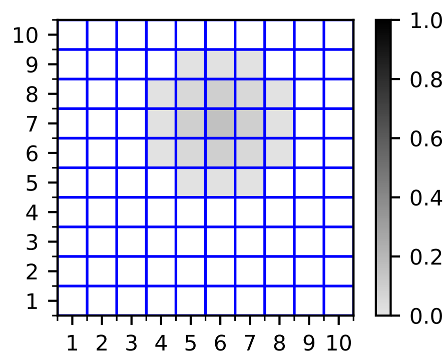


Figure 4.3: Sample two-dimensional probability distribution. In this example, shading is used to indicate the probability of the robot being located in a particular grid cell.

variable representing the output of a wall detection sensor that is designed to tell us when we are near one of the two boundaries of the hallway (state 1 or state 10). This is activated with probability .8 when the robot is at either end of the hallway, .1 when it is one step away from either end and 0 for all other locations.

In the case where we know nothing about the location of the robot, our joint probability distribution will look like the following for our location/sensor scenario:

X	Z	P(X,Z)
1	$beep$	.08
1	$\neg beep$	.02
2	$beep$	.01
2	$\neg beep$	.09
3	$beep$	0
3	$\neg beep$	.1
	...	
8	$beep$	0
8	$\neg beep$	.1
9	$beep$	.01
9	$\neg beep$	.09
10	$beep$	.08
10	$\neg beep$	.02

Table 4.1: Joint probability distribution for robot position X and wall sensor output Z.

Marginalization

Given the full joint distribution we can always recover the probability distribution for an individual variable through **marginalization**, or summing out. In the general case, this can be expressed as:

$$P(A=a) = \sum_{b \in B} P(A=a,\ b) \tag{4.1}$$

This means that if we want to calculate the probability of some assignment to A using the joint distribution, we just need to sum up all of the rows in the table that match that assignment to A. For example, given our location/sensor example above, we can recover the probability that the robot is at position 2 as follows:

$$\begin{aligned} P(X=2) &= P(X=2,\ beep) + P(X=2,\ \neg beep) \\ &= .01 + .09 \\ &= .1 \end{aligned}$$

Stop and Think

4.1 Based on the joint probability distribution in Table 4.1, what is $P(beep)$? What is $P(\neg beep)$? ■

Independence

Two random variables are defined to be **independent** if and only if

$$P(A \cap B) = P(A)P(B). \tag{4.2}$$

Intuitively, A and B are independent if knowing the value of A provides no information about the value of B. For example, whether or not a region will experience an earthquake on a particular day is independent of the occurrence of a tornado: there is no reason to believe that one will make the other more or less likely. On the other hand, whether a region will experience an earthquake on a given day is *not* independent of the possibility of a tsunami: earthquakes can cause tsunamis, so knowing that one has occurred increases the probability of the other.

The notion of independence is important for probabilistic reasoning. In the example from the previous section, we saw a joint probability distribution with two random variables. In general, applications involving probabilistic reasoning may involve *many* random variables. For example, consider a robot navigating through an office building with 20 doors that may each be open or closed. Assuming that the robot is unable to open doors, path planning in this environment will require reasoning about the possible states of all doors. In order to write down the full joint probability distribution describing every possible combination of closed and open doors, we would need a table with $2^{20} \approx 1,000,000$ rows. It is much more efficient, in terms of both space and computation, to make the assumption that the state of each door is independent of the state of every other door. In that case we only need to store 20 individual probability distributions with two rows each. We can then reconstruct the probability of any combination of closed and open doors by simply multiplying together the appropriate 20 probabilities.

Note that **independence assumptions** of this sort may be useful even if they aren't entirely correct. In a real office building it is unlikely that the states of individual doors will be entirely independent. If the doorway to a suite of offices is open, there is probably someone working in that suite, which would make it more likely that the individual office doors will be open as well. This is a case where it is necessary to settle for an approximately correct answer in the interest of computational tractability.

Stop and Think

4.2 The probability that it will rain today, $P(R)$ is independent of the probability of an earthquake, $P(E)$. Assuming $P(r) = .7$ and $P(e) = .001$, what is the probability that it will be a rainy day with no earthquake? $P(r, \neg e) =??$ ■

4.2.2 Conditional Probabilities

Probability distributions like $P(X)$ are sometimes described as **prior probabilities** because they describe our initial belief about X before we have taken any evidence into account. **Conditional probabilities** allow us to express the probability for one random variable, given that we know the value of some other random variable. Conditional probability is defined as follows:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \tag{4.3}$$

Conditional probabilities provide a useful way of thinking about the relationship between a robot's state and sensor data. For robot localization, some variables represent unknown state information that we are attempting to estimate. This unknown state information is commonly denoted X. On the other hand, some variables represent *known* values received from a sensor. These values are commonly denoted Z. For the purpose of localization, it would be useful to have access to the conditional probability distribution $P(X \mid Z)$. This is exactly the distribution over X given our known sensor value Z.

As an example of conditional probability, consider the case of a cliff detection sensor designed to prevent a home-vacuuming robot from falling down stairways. In this case, S is a Boolean random variable indicating whether a stairway is actually present, and the variable Z is true if the cliff-detection sensor has activated. The following table describes the full joint probability distribution:

S	Z	$P(S,Z)$
T	T	.0495
T	F	.0005
F	T	.095
F	F	.855

Using this table, along with Equations 4.1 and 4.3, we can calculate, for example, the false positive rate of our sensor: $P(Z = True \mid S = False)$. In other words, what is the probability that the sensor will indicate that a stairway is present when it is not.

$$\begin{aligned}
P(Z = True \mid S = False) &= \frac{P(Z = True \cap S = False)}{P(S = False)} \\
&= \frac{.095}{P(S = False)} && \text{(From the third row of the table above)} \\
&= \frac{.095}{P(S = False, Z = True) + P(S = False, Z = False)} && \text{(Marginalization)} \\
&= \frac{.095}{.095 + .855} = .1
\end{aligned}$$

Stop and Think

4.3 Look back at table 4.1. Apply definition 4.3 to calculate $P(X = 0 \mid Z = beep)$. ■

Conditional Independence

Section 4.2.1 introduced the idea of independent random variables. Now that we have an understanding of conditional distributions we can introduce a second notion of independence that is also a valuable tool in probabilistic reasoning.

Conditional independence is defined as follows:

A random variable A is conditionally independent of C given B if and only if

$$P(A \mid B, C) = P(A \mid B). \tag{4.4}$$

Intuitively, this means that, if we already know B, then learning something about C doesn't tell us anything additional about A. This is a bit more subtle than the definition of independence introduced earlier. Stating that A is conditionally independent from C given B is not the same as saying that any two of those variables are independent. As an example, consider the following scenario:

- A: The east door to the mall is locked
- B: The mall is open for business
- C: The west door to the mall is locked

In this case, A and C are not independent. If I learn that one door is locked, that increases my belief that the other will be locked as well. However, they *are* conditionally independent given B: if I already know whether or not the mall is open, then learning the state of the west door doesn't give me any additional information about east door.

4.2.3 Bayes' Rule

Unfortunately definition (4.3) often isn't useful for determining $P(X \mid Z)$ because it requires us to have access to the full joint probability distribution $P(X,Z)$. In practice, that full distribution is usually not easy to obtain in an explicit form. **Bayes' rule** or **Bayes' theorem** is tremendously useful here. Baye's rule can be expressed as follows:

$$P(X \mid Z) = \frac{P(Z \mid X)P(X)}{P(Z)} \tag{4.5}$$

Bayes' rule provides a recipe for taking the prior probability $P(X)$ that describes our initial beliefs about the robot's state and calculating a **posterior probability** $P(X \mid Z)$ that describes our updated belief after into account information from a sensor reading.

In order to apply Bayes' rule, we need to know two things:

- $P(Z \mid X)$ - The conditional probability of sensor readings given state information. This is often described as the **sensor model**. It simply captures what we know about how our sensor works.
- $P(X)$ - The prior distribution over our state variable.

It turns out we don't need to know $P(Z)$ explicitly. It can be calculated from $P(Z \mid X)$ and $P(X)$ using the **total probability formula**:

$$P(Z) = \sum_{x \in X} P(Z \mid X = x)P(X = x) \tag{4.6}$$

Alternatively, it is common to to replace $P(Z)$ using a **normalizing constant** α defined as

$$\alpha = \frac{1}{P(Z)}.$$

This results in the following alternate form of Bayes' rule:

$$P(X \mid Z) = \alpha P(Z \mid X)P(X) \tag{4.7}$$

This formulation takes advantage of the fact that we know that $P(Z)$ is the same for every state. We can simply solve for the value of α that makes the posterior probability sum to one.

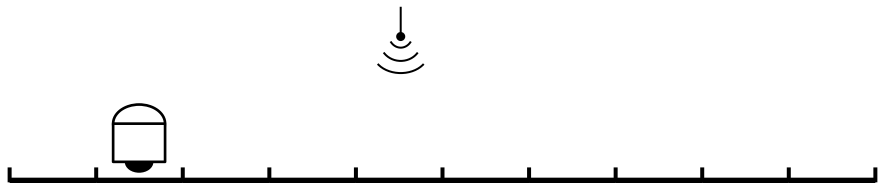


Figure 4.4: A scenario with ten discrete robot locations. There is a radio beacon at location 5.

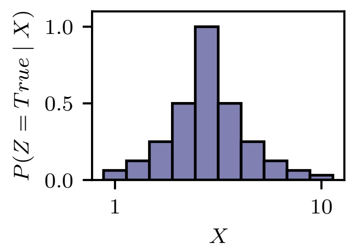


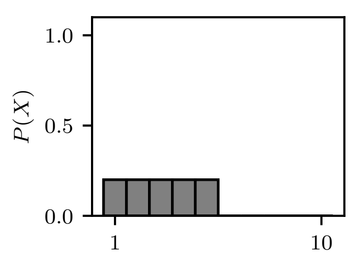
Figure 4.5: Sensor model for our radio beacon detector. The probability that the sensor will activate falls off as the robot moves further away from the beacon. Notice that this figure is *not* illustrating a probability distribution of the type shown in Figure 4.1. In this case the probabilities don't need to sum to one.

Bayes' Rule Example

Consider another one-dimensional robot that may be in one of ten discrete locations. In this case, there is a radio beacon located at location **5** and the robot has a sensor that detects the beacon with a probability that is related to distance. The scenario is illustrated in Figure 4.4.

When the robot is in the same location as the beacon, it will be detected with a probability of 1. When the robot is one step away, it will be detected with a probability of .5. The probability of detection continues to drop off at a rate of 50% for each step away from the beacon. Figure 4.5 illustrates this sensor model.

In this scenario, Bayes' rule allows us to update our belief about where the robot is located based on the signal received from the radio sensor. Let's assume that our prior probability distribution over the robot's location looks like this:



This prior distribution $P(X)$ reflects that robot is equally likely to be in any of the five locations on the left, with a .2 probability of being in each.

Now assume that the robot takes a reading from its beacon-detector and the value is $True$. Figure 4.6 illustrates the steps of applying Bayes' rule to calculate the posterior distribution. First, the prior probability associated with each state $P(x_i)$ is multiplied by the value of the sensor model at that state

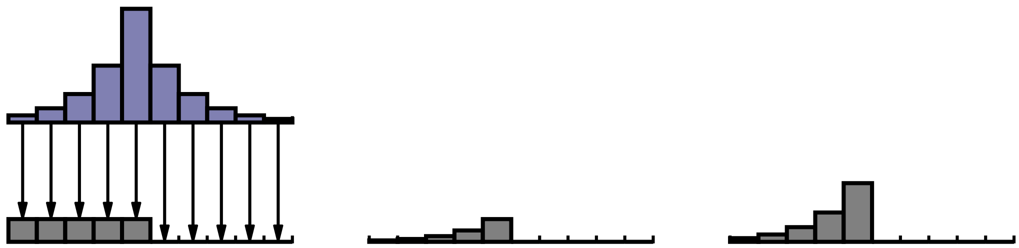


Figure 4.6: Using Bayes' rule to update state estimates based on a sensor reading. (a) The bar graph on the top represents the sensor model $P(Z = True \mid X)$. The bar chart on the bottom represents the prior distribution over the robot's location $P(x)$ The vertical arrows represent multiplication: the value of the sensor model at each state is multiplied by the prior probability for the corresponding state. (b) The resulting products. Note that these values do *not* sum to one. (c) The values at each state re-scaled (or **normalized**) to make the total sum to 1. The result is the posterior probability distribution $P(X \mid Z)$

$P(Z = True \mid X = x_i)$. The resulting values are then re-scaled to sum to one.

Let's do a detailed run-through through of the example in Figure 4.6 to see how the steps follow from the application of Bayes' rule. Bayes' rule may be used to calculate the posterior probability that the robot is in any particular state. For example, calculating the posterior for state 1 looks like the following:

$$P(X = 1 \mid Z = True) = \alpha P(Z = True \mid X = 1)P(X = 1)$$

Substituting $P(Z = True \mid X = 1) = 0.0625$ from the sensor model and $P(X = 1) = .2$ from the prior distribution yields:

$$P(X = 1 \mid Z = True) = \alpha(0.0625 \times .2) = 0.0125\alpha$$

This calculation is repeated for all values of X:

$$\begin{aligned}
P(X = 1 \mid Z = True) &= \alpha(0.0625 \times .2) &= 0.0125\alpha \\
P(X = 2 \mid Z = True) &= \alpha(0.0125 \times .2) &= 0.025\alpha \\
P(X = 3 \mid Z = True) &= \alpha(0.25 \times .2) &= 0.05\alpha \\
P(X = 4 \mid Z = True) &= \alpha(0.5 \times .2) &= 0.1\alpha \\
P(X = 5 \mid Z = True) &= \alpha(1.0 \times .2) &= 0.2\alpha \\
P(X = 6 \mid Z = True) &= \alpha(0.5 \times 0) &= 0 \\
&\ldots
\end{aligned}$$

All that remains is to determine α by solving for the value that makes the total probability sum to 1:

$$0.0125\alpha + 0.025\alpha + 0.05\alpha + 0.1\alpha + 0.2\alpha = 1.0$$

$$\alpha \approx 2.581$$

Substituting 2.581 for α gives us the final posterior probability distribution across states:

$$
\begin{aligned}
P(X=1 \mid Z=True) &= 0.0125\alpha \approx 0.0323 \\
P(X=2 \mid Z=True) &= 0.025\alpha \approx 0.0645 \\
P(X=3 \mid Z=True) &= 0.05\alpha \approx 0.1290 \\
P(X=4 \mid Z=True) &= 0.1\alpha \approx 0.2581 \\
P(X=5 \mid Z=True) &= 0.2\alpha \approx 0.5161 \\
P(X=6 \mid Z=True) &= 0 \\
&\dots
\end{aligned}
$$

4.3 Recursive State Estimation

We now have the tools to formalize the central computational problem raised in this Chapter. The fundamental goal in robot localization is to estimate a robot's position based on the full history of sensor readings and actions taken by the robot. This can be expressed as follows:

$$Bel(X_t) = P(X_t|U_0,Z_0,\ U_1,Z_1,\ \dots,\ U_t,Z_t) \tag{4.8}$$

where the subscripts indicate discrete time intervals and the U variables represent the action choices made by the robot at each time step. According to this definition, the **belief state** $Bel(X_t)$ is defined to be conditional distribution over the robot's location at time step t given full history of actions and sensor readings leading up to time t.

We are already halfway to the solution. At the beginning of this Chapter we claimed that keeping track of a robot's location involves two sources of information: sensor data and dead reckoning. As we've seen above, Bayes' rule exactly solves the problem of incorporating sensor data to update our beliefs about the robot's location.

In order to incorporate dead reckoning we need a probabilistic model of how the robot's actions impact its state. This can be expressed with the following conditional probability distribution:

$$P(X_t \mid X_{t-1}, U_t)$$

This distribution is often described as the **motion model** for the robot. Notice that this formulation involves an implicit conditional independence assumption in that the state distribution at time step t only depends on the most recent state and action. Expressed formally, we are assuming that:

$$P(X_t \mid X_{t-1}, U_t) = P(X_t \mid X_{t-1}, U_t, X_{t-2}, U_{t-1}, \dots, X_0, U_1) \tag{4.9}$$

This is called the **Markov assumption**. This assumption may be expressed in English as "The future is independent of the past given the present."

The Markov assumption is key to developing a tractable algorithm for tracking the belief state. At first glance, the formulation of the localization problem in Equation 4.8 should be worrying. The history of sensor readings and action choices will continue to grow without bound over time. This raises the concern that the computational cost of maintaining our belief state will also continue to grow as we accumulate more and more history that needs to taken into account. The Markov assumption provides a way out of that dilemma.

The full **recursive state estimation** algorithm can be expressed as follows:

$$Bel^-(X_t) = \sum_{x_i \in X_{t-1}} P(X_t \mid X_{t-1} = x_i,\, U_t) Bel(X_{t-1} = x_i) \quad \text{(Prediction)} \tag{4.10}$$

$$Bel(X_t) = \alpha P(Z_t \mid X_{t-1}) Bel^-(X_t) \quad \text{(Correction)} \tag{4.11}$$

In the **prediction** stage, we update the belief state based on the robot's latest action by applying the total probability theorem (Equation 4.6) to the motion model to sum across all possible values for the previous state. The $^-$ superscript indicates that the result is the estimated belief state before information about any sensor readings has been incorporated. This is referred to as the "prediction" step because we are predicting where the robot is likely to end up based on the action it selected.

In the **correction** stage we apply Bayes' rule to update the belief state based on the latest sensor reading, exactly as was discussed in the previous section. This is referred to as the "correction" step because we are revising our prediction by incorporating the latest sensor information.

These two stages alternate indefinitely, with the belief state calculated at each time step serving as the prior belief state for the next time step. Assuming that the Markov assumption is correct, and that the motion and sensor models are accurate, this algorithm correctly tracks the probability distribution over the robot's location over time.

One thing to keep in mind is that a perfect localization algorithm doesn't give us perfect information about the location of the robot. Our knowledge of the robot's location is fundamentally limited by the uncertainty in our sensor model, the uncertainty in our motion model, and our uncertainty about the robot's initial location when localization began. What we can guarantee is that the recursive state estimation algorithm above gives the best possible estimate given these various sources of uncertainty.

Prediction Example

Figure 4.7 illustrates the process of applying the prediction formula. In the motion model for this example, there are three possible outcomes when the robot attempts to move to the right: There is a 60% chance that the action works as expected and the robot moves one cell to the right. There is a 20% chance that the action fails and the robot stays in the same location, and there is a 20% chance that the robot overshoots and moves two positions to the right.

Figure 4.7(a) illustrates the calculation for just state 5. Given this motion model, there are three possible ways the robot could end up in state 5: it could start in state 3 and overshoot, it could start in state 4 with the expected outcome, or it could start in state 5 and fail to move. The prediction formula sums across these three possibilities:

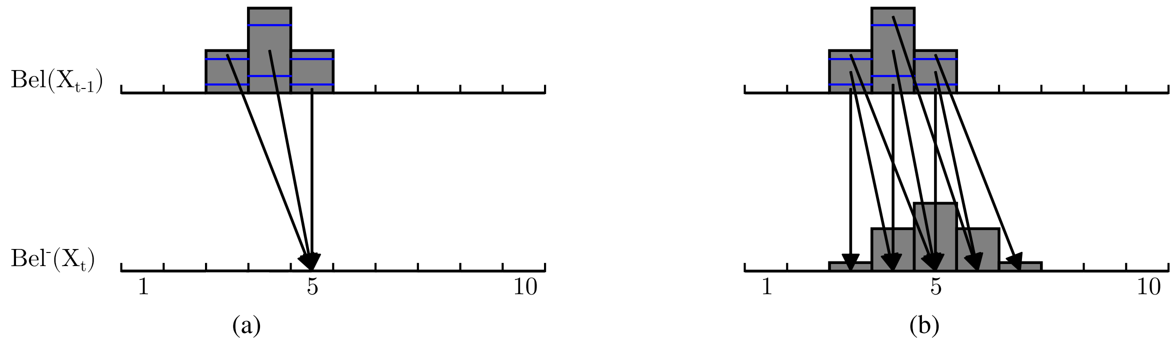


Figure 4.7: Using the prediction formula to update the belief state based on the motion model. (a) Each arrow represents a term in the sum for $Bel^-(X_t = 5)$. (b) The final belief distribution after all sums have been calculated.

$$
\begin{aligned}
Bel^-(X_t = 5) = &\sum_{x_i \in X_{t-1}} P(X_t = 5 \mid X_{t-1} = x_i,\, U_t = Right) Bel(X_{t-1} = x_i) \\
=&P(X_t = 5 \mid X_{t-1} = 3,\, U_t = Right) Bel(X_{t-1} = 3) + \\
&P(X_t = 5 \mid X_{t-1} = 4,\, U_t = Right) Bel(X_{t-1} = 4) + \\
&P(X_t = 5 \mid X_{t-1} = 5,\, U_t = Right) Bel(X_{t-1} = 5) \\
=&(.2 \times .25) + (.6 \times .5) + (.2 \times .25) \\
=&.4
\end{aligned}
$$

The full belief prediction update involves repeating this calculation for each state. The result is illustrated in figure 4.7(b)

4.3.1 Efficiency Considerations

The discrete, grid-based state representation assumed so far raises some of the same efficiency issues that were discussed in Chapter 3. In particular, the space and computational requirements grow exponentially with the dimensionality of the robot's state. Two-dimensional localization may be tractable, but once we add orientation and three-dimensional location, the problem quickly becomes un-manageable. There is also a trade-off between the granularity of the discretization and the precision of localization. Finer granularity may result in better localization, but only at the expense of additional computational cost.

In practice, it is common to use one of two alternative formulations of the recursive state estimation algorithm described above: the **Kalman filter** or the **particle filter**.

4.4 Kalman Filter

The overall form of the Kalman filter algorithm is the same as the recursive state estimation algorithm described above: There is a prediction phase in which the state estimate is projected forward through time, followed by a correction phase in which the state estimate is updated according to the latest sensor reading. The difference is that the Kalman filter uses an alternative representation of the belief state, and makes some strong assumptions about the form of the motion and sensor models:

- The belief state is represented as a multi-variate normal distribution.
- The state update and sensor models must be linear.
- The noise in both the motion and sensor models must be described by multi-variate normal distributions.

If these assumptions are satisfied, the Kalman filter provides an efficient and optimal localization algorithm.

4.4.1 Linear State Dynamics

As an example, consider the following system of difference equations that describe the motion of an object moving at a fixed velocity in two dimensions:

$$\begin{aligned}
x_{t+1} &= x_t + \dot{x}_t dt \\
y_{t+1} &= y_t + \dot{y}_t dt \\
\dot{x}_{t+1} &= \dot{x}_t \\
\dot{y}_{t+1} &= \dot{y}_t
\end{aligned}$$

In these equations x_t and y_t represent the x and y coordinates of the object at time t, $\dot{x}_t$ and $\dot{x}_t$ represent the instantaneous velocity, and dt represents the time interval between updates. The first two equations describe the motion of the object, while the second two equations simply describe the fact that the velocity remains constant over time.

This system can be described more concisely by representing the state information as a vector, and the series of difference equations as the matrix product:

$$\mathbf{x}_{t+1} = \mathbf{F}\mathbf{x}_t$$

Where $\mathbf{x}_t$ represents the state vector and $\mathbf{F}$ is a matrix that describes the update equations:

$$\mathbf{x}_t = \begin{bmatrix} x_t \\ y_t \\ \dot{x}_t \\ \dot{y}_t \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} 1 & 0 & dt & 0 \\ 0 & 1 & 0 & dt \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We may also want to represent the fact that there it is possible to apply a control signal to the object. For example, we could update our update equations with terms representing an application of force that impacts the objects velocity:

$$\begin{aligned}
x_{t+1} &= x_t + \dot{x}_t dt \\
y_{t+1} &= y_t + \dot{y}_t dt \\
\dot{x}_{t+1} &= \dot{x}_t + \ddot{x}_t dt \\
\dot{y}_{t+1} &= \dot{y}_t + \ddot{y}_t dt
\end{aligned}$$

We can incorporate this control signal by adding an additional term to our linear state update equation:

$$\mathbf{x}_{t+1} = \mathbf{F}\mathbf{x}_t + \mathbf{B}\mathbf{u}_t$$

where

$$\mathbf{u}_t = \begin{bmatrix} \ddot{x}_t \\ \ddot{y}_t \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ dt & 0 \\ 0 & dt \end{bmatrix}.$$

Finally, we need to take into account that the motion model should not be completely deterministic. We don't expect any model to make perfect predictions in a real-world system. The Kalman filter works under the assumption that system noise is normally distributed. We can incorporate noise by adding one final term to our state update equation:

$$\mathbf{x}_t = \mathbf{F}\mathbf{x}_{t-1} + \mathbf{B}\mathbf{u}_{t-1} + \mathbf{w}_{t-1} \tag{4.12}$$

Where $\mathbf{w}$ is a random vector drawn from a normal distribution with mean zero and covariance Q:

$$\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q})$$

4.4.2 Linear Sensor Model

The sensor model for a Kalman filter is also represented by a linear model corrupted with normally-distributed noise. The sensor model has the following form:

$$\mathbf{z}_t = \mathbf{H}\mathbf{x}_t + \mathbf{v}_t \tag{4.13}$$

Where $\mathbf{z}_t$ represents the observed sensor reading and $\mathbf{v}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{R})$ represents normally distributed sensor noise with covariance $\mathbf{R}$.

Continuing the example from above, we will assume that we have access to a sensor that provides estimates about the coordinates of the object, but doesn't provide any direct information about the velocity. In this case, we have

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

4.4.3 Kalman Filter Algorithm

The Kalman filter stores the belief state as a normal distribution. The mean of the distribution $\hat{\mathbf{x}}$ represents the current best estimate of the system state, while the covariance $\mathbf{P}$ represents the amount (and shape) of the current uncertainty. As with the recursive state estimation algorithm described in Section 4.3, the algorithm proceeds in two stages: a prediction stage, followed by a correction stage. Again, the $^-$ superscript indicates that the result is the estimated belief state before sensor readings have been incorporated. The full algorithm is outlined in Figure 4.8.

Inputs: Initial state estimate $\hat{\mathbf{x}}_0$ and covariance $\mathbf{P}_0$.

- **Repeat forever:**
  - **Prediction**
    - Project the state forward according to the motion model:

$$\hat{\mathbf{x}}_t^- = F\hat{\mathbf{x}}_{t-1} + B\mathbf{u}_{t-1} \tag{4.14}$$

    - Project the covariance of the state estimate forward:

$$\mathbf{P}_t^- = F\mathbf{P}_{t-1}F^T + \mathbf{Q} \tag{4.15}$$

  - **Correction**
    - Compute the Kalman gain:

$$\mathbf{K}_t = \mathbf{P}_t^- H^T (H\mathbf{P}_t^- H^T + \mathbf{R})^{-1} \tag{4.16}$$

    - Use the sensor reading to update the state estimate:

$$\hat{\mathbf{x}}_t = \hat{\mathbf{x}}_t^- + \mathbf{K}_t(\mathbf{z}_t - H\hat{\mathbf{x}}_t^-) \tag{4.17}$$

    - Update the covariance of the state estimate:

$$\mathbf{P}_t = \mathbf{P}_t^- - \mathbf{K}_t H \mathbf{P}_t^- \tag{4.18}$$

Figure 4.8: Kalman filter algorithm.

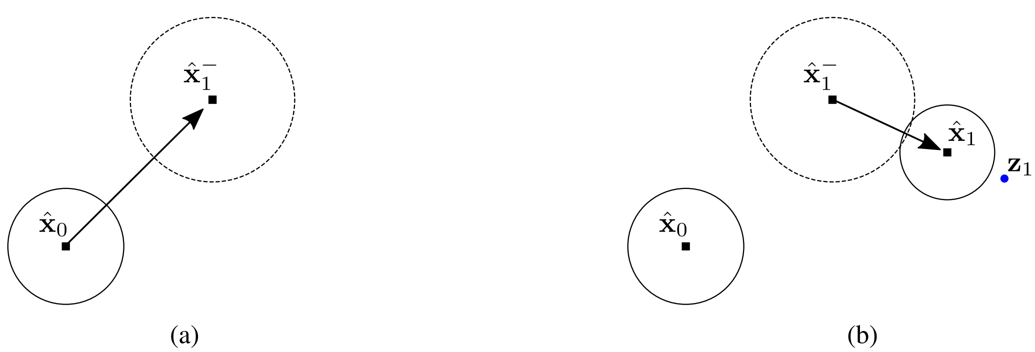


Figure 4.9: Kalman filter example. (a) **Prediction phase:** The state estimate is updated according to the linear motion model. The solid circle represents the initial uncertainty, while the dotted circle represents the increased uncertainty based on the known noise in the motion model. (b) **Correction phase:**. The state prediction is adjusted in the direction of the sensor reading. The uncertainty in the estimate decreases as a result of incorporating the sensor information.

The prediction phase of the algorithm updates the state estimate according to the motion model (Equation 4.14) and increases the estimated uncertainty according the noise in the motion model (Equation 4.15). This is illustrated in 4.9.

The correction phase involves updating the state estimate to account for the most recent sensor reading. Equation 4.17 can be understood as taking a weighted average between the prediction estimate and the estimate that is suggested by the latest sensor value. The **Kalman gain**, calculated in Equation 4.16, represents the optimal trade-off between those two sources of information. Intuitively, the Kalman filter places a higher weight on the sensor when the sensor noise is low relative to the uncertainty in the state estimate. Conversely, the Kalman filter places less weight on the sensor reading if the sensor noise is high relative to the current uncertainty. Put another way: the Kalman filter puts more trust in the more reliable source of information.

4.5 References and Further Reading

The mathematical foundations of probability theory are too broad and varied to survey here. Many of the key formalisms were developed by Pierre-Simon Laplace in the early 19th century.

General artificial intelligence textbooks such as (Russell & Norvig, 2010) and (Poole & Mackworth, 2017) provide an introduction to probabilistic modeling and reasoning as computational tools. The standard reference for the application of probabilistic algorithms in robotics is (Thrun et al., 2005), which provides much more detail on all of the algorithms discussed in this chapter.

The Kalman filter was introduced by R.E. Kalman in 1960 (Kalman, 1960). Greg Welch and Gary Bishop published a popular tutorial introduction that provides a concise overview of the algorithm along with a derivation (Welch & Bishop, 1995).

A history and overview of the particle filtering algorithm is provided by (Godsill, 2019), which attributes key steps in its development to Adrian Smith's research group at Imperial College London in the 1990's (Smith & Gelfand, 1992).