Representing Graphs and Graph Isomorphism

Section 10.3

Representing Graphs: Adjacency Lists

Definition: An *adjacency list* can be used to represent a graph with no multiple edges by specifying the vertices that are adjacent to each vertex of the graph.

Example:

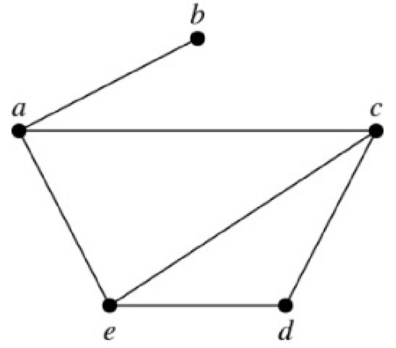


TABLE 1 An Adjacency List for a Simple Graph.

Vertex	*Adjacent Vertices*
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

Example:

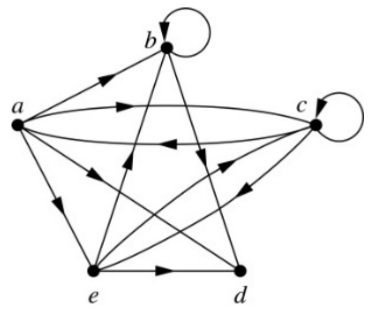


TABLE 2 An Adjacency List for a Directed Graph.

Initial Vertex	*Terminal Vertices*
a	b, c, d, e
b	b, d
c	a, c, e
d	
e	b, c, d

Representation of Graphs: Adjacency Matrices

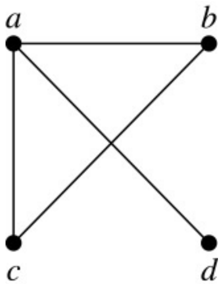
Definition: Suppose that $G = (V, E)$ is a simple graph where $|V| = n$. Arbitrarily list the vertices of G as $v_1, v_2, \ldots, v_n$. The *adjacency matrix* $\mathbf{A}_G$ of G, with respect to the listing of vertices, is the $n \times n$ zero-one matrix with 1 as its (i, j) entry when v_i and v_j are adjacent, and 0 as its (i, j) entry when they are not adjacent.

- In other words, if the graphs adjacency matrix is $\mathbf{A}_G = [a_{ij}]$, then

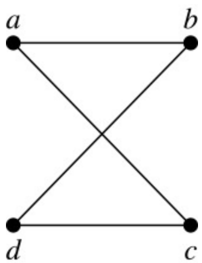
$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

Adjacency Matrices (*continued*)

Example:



$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$



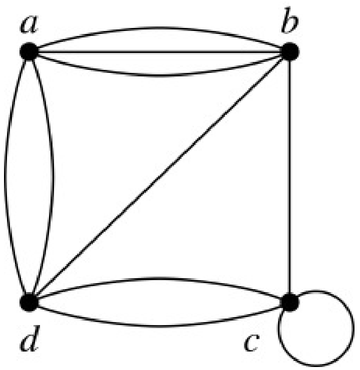
$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

When a graph is sparse, that is, it has few edges relatively to the total number of possible edges, it is much more efficient to represent the graph using an adjacency list than an adjacency matrix. But for a dense graph, which includes a high percentage of possible edges, an adjacency matrix is preferable.

Note: The adjacency matrix of a simple graph is symmetric, i.e., $a_{ij} = a_{ji}$
Also, since there are no loops, each diagonal entry a_{ij} for i = 1, 2, 3, …, n, is 0.

Adjacency Matrices (*continued*)

- Adjacency matrices can also be used to represent graphs with loops and multiple edges.



$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

Adjacency Matrices (*continued*)

- Adjacency matrices can also be used to represent directed graphs.
- The matrix for a directed graph $G = (V, E)$ has a 1 in its (i, j) position if there is an edge from v_i to v_j, where $v_1, v_2, \ldots v_n$ is a list of the vertices.
  - In other words, if the graphs adjacency matrix is $\mathbf{A}_G = [a_{ij}]$, then

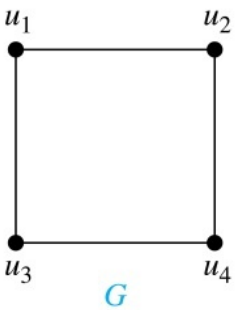
$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

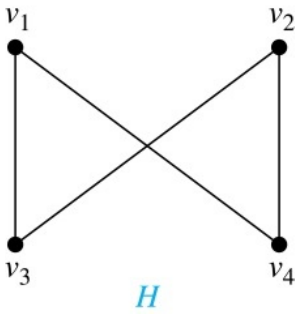
Isomorphism of Graphs

Definition: The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there is a one-to-one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such a function f is called an *isomorphism.* Two simple graphs that are not isomorphic are called *nonisomorphic*.

Isomorphism of Graphs (*cont.*)

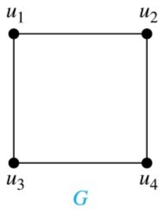
Example: Show that the graphs $G = (V, E)$ and $H = (W, F)$ are isomorphic.

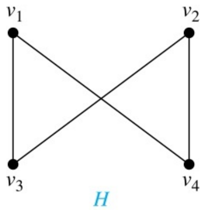




Isomorphism of Graphs (*cont.*)

Example: Show that the graphs $G = (V, E)$ and $H = (W, F)$ are isomorphic.





Solution: The function f with $f(u_1) = v_1$,

$f(u_2) = v_4$, $f(u_3) = v_3$, and $f(u_4) = v_2$ is a

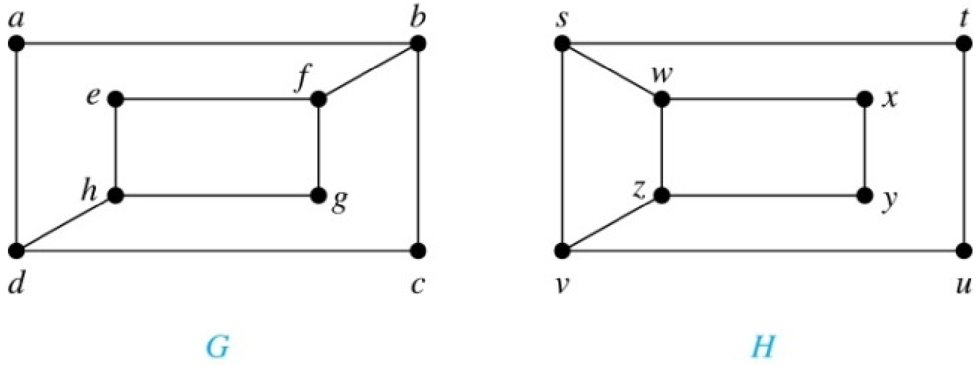
one-to-one correspondence between V and W with the necessary adjacencies.

Isomorphism of Graphs (*cont.*)

- It is difficult to determine whether two simple graphs are isomorphic using brute force because there are $n!$ possible one-to-one correspondences between the vertex sets of two simple graphs with n vertices.
- The best algorithms for determining weather two graphs are isomorphic have exponential worst case complexity in terms of the number of vertices of the graphs.
- Sometimes it is not hard to show that two graphs are not isomorphic. We can do so by finding a property, preserved by isomorphism, that only one of the two graphs has. Such a property is called *graph invariant*.
- Useful graph invariants:
  - number of vertices,
  - number of edges,
  - degree sequence (list of the degrees of the vertices in nonincreasing order).

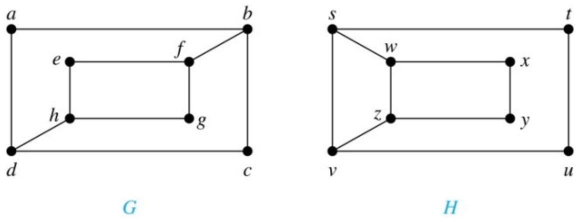
Isomorphism of Graphs

Example: Determine whether these two graphs are isomorphic.



Isomorphism of Graphs

Example: Determine whether these two graphs are isomorphic.



Solution: Both graphs have eight vertices and ten edges.

They also both have four vertices of degree two and four of degree three.

However, *G* and *H* are not isomorphic. Note that since *deg*(*a*) = 2 in *G*, *a* must correspond to *t*, *u*, *x*, or *y* in H, because these are the vertices of degree 2. But each of these vertices is adjacent to another vertex of degree two in *H*, which is not true for *a* in *G*.