CS228 - Basic Counting and the Pigeonhole Principle

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Material in these slides is from "Discrete Mathematics and Its Applications 7e", Kenneth Rosen, 2012.

The Product Rule

A procedure can be broken down into a sequence of two tasks. There are n_1 ways to do the first task and for each of these ways there are n_2 ways to do the second task. Then there are $n_1 \cdot n_2$ ways to do the procedure.

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Example: How many bit strings of length seven are there?.
Solution: Each bit is either 0 or 1, so the answer is $2^7 = 128$

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How many *one to one* functions are there from a set with m elements to a set with n elements?

Solution: $n(n-1)(n-2)\ldots(n-m+1)$ such functions.

Product Rule in Terms of Sets

If $A_1, A_2, \ldots, A_m$ are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements of each set. The task of choosing an element in the Cartesian product $A_1 \times A_2 \times \ldots \times A_m$ is done by choosing an element in A_1, an element in A_2 , ... , and an element in A_m. By the product rule, it follows that:

$$|A_1 \times A_2 \times \ldots \times A_m| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_m|$$

The Sum Rule

If a task can be done either in one of n_1 ways or in one of n_2 ways to do the second task, where none of the set of n_1 ways is the same as any of the n_2 ways, then there are $n_1 + n_2$ ways to do the task.

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Example: The mathematics department must choose either a student or a faculty to serve on a committee. How many choices are there for this representative if there are 37 faculty members and 83 majors and no one is both a faculty member and a student.

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Example: The mathematics department must choose either a student or a faculty to serve on a committee. How many choices are there for this representative if there are 37 faculty members and 83 majors and no one is both a faculty member and a student.

Solution: By the sum rule it follows that there are $37 + 83 = 120$ possible ways to pick a representative.

The Sum Rule in Terms of Sets

$|A \cup B| = |A| + |B|$ as long as A and B are disjoint sets.

More generally:

$$|A_1 \cup A_2 \cup \ldots \cup A_m| = |A_1| + |A_2| + \ldots + |A_m|$$

as long as $A_i \cap A_j = \emptyset$ for all i, j.

The Subtraction Rule

If a task can be done either in one of n_1 ways or in one of n_2 ways, then the total number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

Also known as the *principle of inclusion-exclusion*:

$|A \cup B| = |A| + |B| - |A \cap B|$

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Solution:

- Number of bit strings of length 8 that start with a 1: $2^7 = 128$.
- Number of bit strings that end with 00: $2^6 = 64$.
- Number of bit strings that start with 1 and end with 00: $2^5 = 32$.
- Answer: $128 + 64 - 32 = 160$.

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If k is a positive integer and $k + 1$ objects are placed into k boxes, then at least one box contains two or more objects.

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Example: Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

Generalized Pigeonhole Principle

If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Example: Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.